

# A GIT INTERPRETATION OF THE HARDER-NARASIMHAN FILTRATION

TOMÁS L. GÓMEZ, IGNACIO SOLS AND ALFONSO ZAMORA

**ABSTRACT.** An unstable torsion free sheaf on a smooth projective variety gives a GIT-unstable point in certain Quot scheme. To a GIT-unstable point, Kempf associates a “maximally destabilizing” 1-parameter subgroup, and this induces a filtration of the torsion free sheaf. We show that this filtration coincides with the Harder-Narasimhan filtration. Then we prove the analogous result for holomorphic pairs.

## INTRODUCTION

Let  $X$  be a smooth complex projective variety, and let  $\mathcal{O}_X(1)$  be an ample line bundle on  $X$ . If  $E$  is a coherent sheaf on  $X$ , let  $P_E$  be its Hilbert polynomial with respect to  $\mathcal{O}_X(1)$ , i.e.,  $P_E(m) = \chi(E \otimes \mathcal{O}_X(m))$ . If  $P$  and  $Q$  are polynomials, we write  $P \leq Q$  if  $P(m) \leq Q(m)$  for  $m \gg 0$ .

A torsion free sheaf  $E$  on  $X$  is called semistable if for all proper subsheaves  $F \subset E$ ,

$$\frac{P_F}{\mathrm{rk} F} \leq \frac{P_E}{\mathrm{rk} E}.$$

If it is not semistable, it is called unstable, and it has a canonical filtration:

Given a torsion free sheaf  $E$ , there exists a unique filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E,$$

which satisfies the following properties, where  $E^i := E_i/E_{i-1}$ :

- (1) Every  $E^i$  is semistable.
- (2) The Hilbert polynomials verify

$$\frac{P_{E^1}}{\mathrm{rk} E^1} > \frac{P_{E^2}}{\mathrm{rk} E^2} > \cdots > \frac{P_{E^{t+1}}}{\mathrm{rk} E^{t+1}}$$

This filtration is called the *Harder-Narasimhan filtration* of  $E$  ([HL2, Theorem 1.3.6]).

We will briefly describe the construction of the moduli space for these objects. This is originally due to Gieseker for surfaces, and it was generalized to higher dimension by Maruyama ([Gi, Ma]). To construct the moduli space of torsion free sheaves with fixed Hilbert polynomial  $P$ , we choose a suitably large integer  $m$  and consider the Quot scheme parametrizing quotients

$$(0.1) \quad V \otimes \mathcal{O}_X(-m) \longrightarrow E$$

where  $V$  is a fixed vector space of dimension  $P(m)$  and  $E$  is a sheaf with  $P_E = P$ . The Quot scheme has a canonical action by  $\mathrm{SL}(V)$ . Gieseker [Gi] gives a linearization of this action on a certain ample line bundle, in order to use Geometric Invariant Theory to take the quotient by the action. The moduli space of semistable sheaves is obtained as the GIT quotient.

Let  $E$  be a torsion free sheaf which is unstable. Choosing  $m$  large enough (depending on  $E$ ), and choosing an isomorphism  $V \cong H^0(E(m))$ , we obtain a quotient as in (0.1). The corresponding point in the Quot scheme will be GIT-unstable. By the Hilbert-Mumford criterion, there will be a 1-parameter subgroup of  $\mathrm{SL}(V)$  which “destabilizes” the point. Among all these 1-parameter subgroups, Kempf shows that there is a conjugacy class of “maximally destabilizing” 1-parameter subgroups, all of them giving a unique weighted filtration of  $V$ . This filtration induces a sheaf filtration of  $E$ . In principle, this filtration will depend on the integer  $m$ , but we show that it stabilizes for  $m \gg 0$ , and we call it the *Kempf filtration of  $E$* . In this article, we show that the Kempf filtration of an unstable torsion free sheaf  $E$  coincides with the Harder-Narasimhan filtration.

We also prove the analogous result for holomorphic pairs. We remark that the Harder-Narasimhan filtration for holomorphic pairs has been obtained in [Sa].

The equality between the Harder-Narasimhan filtration and the Kempf filtration for torsion free sheaves has been proved by Hoskins and Kirwan [HK] using a different method. One difference with our approach is that they use the existence of the Harder-Narasimhan filtration, whereas we do not use it. Therefore, in principle our method could be used to define a Harder-Narasimhan filtration, using the Kempf filtration, in a moduli problem where there is still no Harder-Narasimhan filtration known. We will come to this application in a future work.

If we replace Hilbert polynomials with degrees, the notion of semistability becomes  $\mu$ -semistability (also known as slope semistability) and we obtain the  $\mu$ -Harder-Narasimhan filtration. In [Br, BT], Bruasse and Teleman give a gauge-theoretic interpretation of the  $\mu$ -Harder-Narasimhan filtration for torsion free sheaves and for holomorphic pairs. They also use Kempf’s ideas, but in the setting of the gauge group, so they have to generalize Kempf’s results to infinite dimensional groups.

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## 1. A THEOREM BY KEMPF

Following the usual convention, whenever “(semi)stable” and “ $(\leq)$ ” appear in a sentence, two statements should be read: one with “semistable” and “ $\leq$ ” and another with “stable” and “ $<$ ”.

Let  $X$  be a smooth complex projective variety of dimension  $n$  endowed with a fixed polarization  $\mathcal{O}_X(1)$ . A torsion free sheaf  $E$  on  $X$  is said to be (semi)stable if for all proper subsheaves  $F$

$$(1.1) \quad \frac{P_F}{\mathrm{rk} F} (\leq) \frac{P_E}{\mathrm{rk} E} .$$

We will recall Gieseker’s construction [Gi] of the moduli space of semistable torsion free sheaves with fixed Hilbert polynomial  $P$  and fixed determinant  $\det(E) \cong \Delta$ .

A coherent sheaf is called  $m$ -regular if  $h^i(E(m-i)) = 0$  for all  $i > 0$ .

**Lemma 1.1.** *If  $E$  is  $m$ -regular then the following holds*

- (1)  *$E$  is  $m'$ -regular for  $m' > m$*

- (2)  $E(m)$  is globally generated
- (3) For all  $m' \geq 0$  the following homomorphisms are surjective

$$H^0(E(m)) \otimes H^0(\mathcal{O}_X(m')) \longrightarrow H^0(E(m+m')) .$$

Let  $m$  be a suitable large integer, so that  $E$  is  $m$ -regular for all semistable  $E$  (c.f. [Ma, Corollary 3.3.1 and Proposition 3.6]). Let  $V$  be a vector space of dimension  $p := P(m)$ . Given an isomorphism  $V \cong H^0(E(m))$  we obtain a quotient

$$q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E ,$$

hence a homomorphism

$$Q : \wedge^r V \cong \wedge^r H^0(E(m)) \longrightarrow H^0(\wedge^r(E(m))) \cong H^0(\Delta(rm)) =: A$$

and points

$$Q \in \text{Hom}(\wedge^r V, A) \quad \overline{Q} \in \mathbb{P}(\text{Hom}(\wedge^r V, A)) ,$$

where  $Q$  is well defined up to a scalar because the isomorphism  $\det(E) \cong \Delta$  is well defined up to a scalar, and hence  $\overline{Q}$  is a well defined point. Two different isomorphisms between  $V$  and  $H^0(E(m))$  differ by the action of an element of  $\text{GL}(V)$ , but, since an homothety does not change the point  $\overline{Q}$ , to get rid of the choice of isomorphism it is enough to take the quotient by the action of  $\text{SL}(V)$ .

A weighted filtration  $(V_\bullet, n_\bullet)$  of  $V$  is a filtration

$$(1.2) \quad 0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_t \subsetneq V_{t+1} = V,$$

and rational numbers  $n_1, n_2, \dots, n_t > 0$ . To a weighted filtration we associate a vector of  $\mathbb{C}^p$  defined as  $\Gamma = \sum_{i=1}^t n_i \Gamma^{(\dim V_i)}$  where

$$(1.3) \quad \Gamma^{(k)} := (\overbrace{k-p, \dots, k-p}^k, \overbrace{k, \dots, k}^{p-k}) \quad (1 \leq k < p) .$$

Hence, the vector is of the form

$$\Gamma = (\overbrace{\Gamma_1, \dots, \Gamma_1}^{\dim V^1}, \overbrace{\Gamma_2, \dots, \Gamma_2}^{\dim V^2}, \dots, \overbrace{\Gamma_{t+1}, \dots, \Gamma_{t+1}}^{\dim V^{t+1}}) ,$$

where  $V^i = V_i/V_{i-1}$ . Giving the numbers  $n_1, \dots, n_t$  is clearly equivalent to giving the numbers  $\Gamma_1, \dots, \Gamma_{t+1}$  because

$$n_i = \frac{\Gamma_{i+1} - \Gamma_i}{p} \quad \text{and} \quad \sum_{i=1}^{t+1} \Gamma_i \dim V^i = 0$$

A 1-parameter subgroup of  $\text{SL}(V)$  (which we denote in the following by 1-PS) is a non-trivial homomorphism  $\mathbb{C}^* \rightarrow \text{SL}(V)$ . To a 1-PS we associate a weighted filtration as follows. There is a basis  $\{e_1, \dots, e_p\}$  of  $V$  where it has a diagonal form

$$t \mapsto \text{diag}(t^{\Gamma_1}, \dots, t^{\Gamma_1}, t^{\Gamma_2}, \dots, t^{\Gamma_2}, \dots, t^{\Gamma_{t+1}}, \dots, t^{\Gamma_{t+1}})$$

with  $\Gamma_1 < \dots < \Gamma_{t+1}$ . Let

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t+1} = V$$

be the associated filtration. Note that two 1-PS give the same filtration if and only if they are conjugate by an element of the parabolic subgroup of  $\text{SL}(V)$  defined by the filtration.

The basis  $\{e_1, \dots, e_p\}$ , together with a basis  $\{w_j\}$  of  $A$ , induces a basis of  $\text{Hom}(\wedge^r V, A)$  indexed in a natural way by tuples  $(i_1, \dots, i_r, j)$  with  $i_1 < \dots < i_r$ , and the coordinate corresponding to such an index is acted by the 1-PS as:

$$Q_{i_1, \dots, i_r, j} \mapsto t^{\Gamma_{i_1} + \dots + \Gamma_{i_r}} Q_{i_1, \dots, i_r, j}$$

The coordinate  $(i_1, \dots, i_r, j)$  of the point corresponding to  $E$  is non-zero if and only if the evaluations of the sections  $e_1, \dots, e_r$  are linearly independent for generic  $x \in X$ . Therefore, the “minimal relevant weight” which has to be calculated to apply Hilbert-Mumford criterion for GIT stability is

$$\begin{aligned} \mu(\overline{Q}, V_\bullet, n_\bullet) &= \min\{\Gamma_{i_1} + \dots + \Gamma_{i_r} : Q_{i_1, \dots, i_r, j} \neq 0\} \\ &= \min\{\Gamma_{i_1} + \dots + \Gamma_{i_r} : Q(e_{i_1} \wedge \dots \wedge e_{i_r}) \neq 0\} \\ (1.4) \quad &= \min\{\Gamma_{i_1} + \dots + \Gamma_{i_r} : e_{i_1}(x), \dots, e_{i_r}(x) \\ &\quad \text{linearly independent for generic } x \in X\} \end{aligned}$$

After a short calculation (originally due to Gieseker) we obtain

$$(1.5) \quad \mu(\overline{Q}, V_\bullet, n_\bullet) = \sum_{i=1}^t n_i (r \dim V_i - r_i \dim V) = \sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V} (r^i \dim V - r \dim V^i)$$

(recall  $n_i = \frac{\Gamma_{i+1} - \Gamma_i}{p}$ ), where  $r_i = \text{rk } E_i$ ,  $r^i = \text{rk } E^i$ , and  $E_i$  is the sheaf generated by evaluation of the sections of  $V_i$  and  $E^i = E_i / E_{i-1}$ .

By the Hilbert-Mumford criterion ([GIT, Theorem 2.1], [Ne, Theorem 4.9]), a point

$$\overline{Q} \in \mathbb{P}(\text{Hom}(\wedge^r V, A))$$

is GIT (semi)stable if and only if for all weighted filtrations

$$\mu(\overline{Q}, V_\bullet, n_\bullet) (\leq) 0.$$

Using the previous calculation, this can be stated as follows:

**Lemma 1.2.** *A point  $\overline{Q}$  is GIT (semi)stable if for all weighted filtrations  $(V_\bullet, n_\bullet)$*

$$\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V) (\leq) 0.$$

A weighted filtration  $(E_\bullet, n_\bullet)$  of a sheaf  $E$  of rank  $r$  is a filtration

$$(1.6) \quad 0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \dots \subsetneq E_t \subsetneq E_{t+1} = E,$$

and rational numbers  $n_1, n_2, \dots, n_t > 0$ . To a weighted filtration we associate a vector of  $\mathbb{C}^r$  defined as  $\gamma = \sum_{i=1}^t n_i \gamma^{(\text{rk } E_i)}$  where

$$\gamma^{(k)} := (\overbrace{k-r, \dots, k-r}^k, \overbrace{k, \dots, k}^{r-k}) \quad (1 \leq k < r).$$

Hence, the vector is of the form

$$\gamma = (\overbrace{\gamma_1, \dots, \gamma_1}^{\text{rk } E^1}, \overbrace{\gamma_2, \dots, \gamma_2}^{\text{rk } E^2}, \dots, \overbrace{\gamma_{t+1}, \dots, \gamma_{t+1}}^{\text{rk } E^{t+1}}),$$

where  $n_i = \frac{\gamma_{i+1} - \gamma_i}{r}$ , and  $E^i = E_i / E_{i-1}$ .

The following theorem follows from [Gi, Ma]

**Theorem 1.3.** *Let  $E$  be a sheaf. There exists an integer  $m_0(E)$  such that, for  $m > m_0(E)$ , the associated point  $\overline{Q}$  is GIT semistable if and only if the sheaf is semistable.*

Let  $E$  be an unstable sheaf. We choose an integer  $m_0$  larger than  $m_0(E)$  and larger than the integer used in Gieseker's construction of the moduli space.

Through Geometric Invariant Theory, stability of a point in the parameter space can be checked by 1-parameter subgroups (c.f. Hilbert-Mumford criterion, Proposition 1.2): a point is unstable if there exists any 1-PS which makes some quantity positive. It is a natural question to ask if there exists a best way of destabilizing a GIT-unstable point, i.e. a best 1-PS which gives maximum in the quantity we referred to. George R. Kempf explores this idea in [Ke] and answers yes to the question, finding that there exists an special class of 1-parameter subgroups which moves most rapidly toward the origin.

We have seen that giving a weighted filtration, i.e. a filtration of vector subspaces  $V_1 \subset \dots \subset V_{t+1} = V$  and rational numbers  $n_1, \dots, n_t > 0$ , is equivalent to giving a parabolic subgroup with weights, which determines uniquely the vector  $\Gamma$  of a one-parameter subgroup and two of these 1-PS are conjugated by the parabolic and come from the same weighted filtration. We define the following function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}},$$

which we call *Kempf function*. Note that  $\mu(V_\bullet, n_\bullet) = \mu(V_\bullet, \alpha n_\bullet)$ , for every  $\alpha > 0$ , hence by multiplying each  $n_i$  by the same scalar  $\alpha$ , which we call *rescaling the weights*, we get another 1-PS but the same value for the Kempf function.

Note that this function corresponds to the one given in [Ke, Theorem 2.2]. The numerator of both functions coincide with the calculation of the minimal relevant weight by Hilbert-Mumford criterion for GIT-stability (c.f. (1.5)), and the denominator is the norm  $\|\Gamma\|$  of the vector

$$\Gamma = (\overbrace{\Gamma_1, \dots, \Gamma_1}^{\dim V^1}, \overbrace{\Gamma_2, \dots, \Gamma_2}^{\dim V^2}, \dots, \overbrace{\Gamma_{t+1}, \dots, \Gamma_{t+1}}^{\dim V^{t+1}})$$

as it is defined in [Ke] as the Killing length of  $\Gamma$ . Recall that for a simple group  $G$  (as it is the case of  $G = \mathrm{SL}(V)$ ) every bilinear symmetric invariant form is a multiple of the Killing form, and the norm  $\|\Gamma\|$  verifies these properties.

We take the GIT quotient by the group  $G = \mathrm{SL}(V)$ , for which, [Ke, Theorem 2.2] states that whenever there exists any  $\Gamma$  giving a positive value for the numerator of the function (i.e. whenever there exists a 1-PS whose minimal relevant weight is positive, which is equivalent to the sheaf  $E$  to be unstable), there exists a unique parabolic subgroup containing a unique 1-parameter subgroup in each maximal torus, giving maximum in the Kempf function i.e., there exists a unique weighted filtration for which the Kempf function achieves a maximum. Note that we divide by the norm in the Kempf function to have  $\mu(V_\bullet, n_\bullet) = \mu(V_\bullet, \alpha n_\bullet)$ ,  $\forall \alpha > 0$ , hence a well defined maximal weighted filtration for the function is defined up to rescaling, i.e., up to multiplying every weight by the same scalar.

Therefore, [Ke, Theorem 2.2] rewritten in our case asserts the following:

**Theorem 1.4** (Kempf). *There exists a unique weighted filtration (i.e.,  $V_1 \subset \dots \subset V_{t+1} = V$  and rational numbers  $n_1, \dots, n_t > 0$ ) up to multiplication by a scalar, called Kempf filtration*

of  $V$ , such that the Kempf function  $\mu(V_\bullet, n_\bullet)$  achieves the maximum among all filtrations and weights  $n_i > 0$ .

We will construct a filtration by subsheaves of  $E$  (which we will call Kempf filtration of  $E$ ) out of the Kempf filtration of  $V$ . Then we will relate the filtration given by Kempf with the filtration constructed by Harder and Narasimhan to conclude that both filtrations are the same.

## 2. CONVEX CONES

In this subsection we define the machinery which will serve us in the following. Endow  $\mathbb{R}^{t+1}$  with an inner product  $(\cdot, \cdot)$  defined by a diagonal matrix

$$\begin{pmatrix} b^1 & & 0 \\ & \ddots & \\ 0 & & b^{t+1} \end{pmatrix}$$

where  $b^i$  are positive integers. Let

$$\mathcal{C} = \{x \in \mathbb{R}^{t+1} : x_1 < x_2 < \cdots < x_{t+1}\},$$

$$\overline{\mathcal{C}} = \{x \in \mathbb{R}^{t+1} : x_1 \leq x_2 \leq \cdots \leq x_{t+1}\},$$

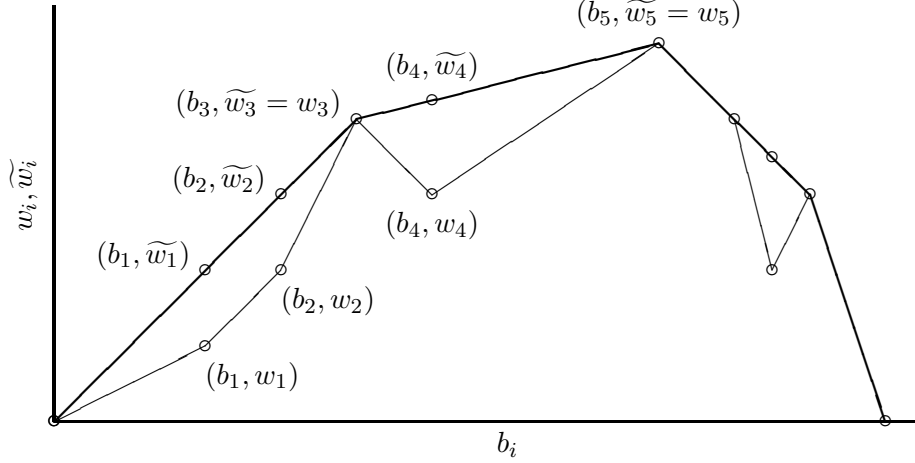
and let  $v = (v_1, \dots, v_{t+1}) \in \mathbb{R}^{t+1} - \{0\}$  verifying  $\sum_{i=0}^{t+1} v_i b^i = 0$ . Define the function

$$\begin{aligned} \mu_v : \overline{\mathcal{C}} - \{0\} &\rightarrow \mathbb{R} \\ \Gamma &\mapsto \mu_v(\Gamma) = \frac{(\Gamma, v)}{\|\Gamma\|} \end{aligned}$$

and note that  $\mu_v(\Gamma) = \|v\| \cdot \cos \beta(\Gamma, v)$ , where  $\beta(\Gamma, v)$  is the angle between  $\Gamma$  and  $v$ . Then, the function  $\mu_v(\Gamma)$  does not depend on the norm of  $\Gamma$  and takes the same value on every point of the ray spanned by each  $\Gamma$ .

Assume that there exists  $\Gamma \in \overline{\mathcal{C}}$  with  $\mu_v(\Gamma) > 0$ . In that case, we want to find a vector  $\Gamma \in \overline{\mathcal{C}}$  which maximizes the function defined before.

Let  $w^i = -b^i v_i$ ,  $w_0 = 0$ ,  $w_i = w^1 + \cdots + w^i$ ,  $b_0 = 0$ , and  $b_i = b^1 + \cdots + b^i$ . Note that  $w_{t+1} = 0$ , by construction. We draw a graph joining the points with coordinates  $(b_i, w_i)$ . Note that this graph has  $t+1$  segments, each segment has slope  $-v_i$  and width  $b^i$ . This is the graph drawn with a thin line in the figure. Now draw the convex envelope of this graph (thick line in the figure), whose coordinates we denote by  $(b_i, \widehat{w}_i)$ , and let us define  $\Gamma_i = -\frac{\widehat{w}_i - \widehat{w}_{i-1}}{b^i}$ . In other words, the quantities  $-\Gamma_i$  are the slopes of the convex envelope graph. We call the vector defined in this way  $\Gamma_v$ . Note that the vector  $\Gamma_v = (\Gamma_1, \dots, \Gamma_{t+1})$  belongs to  $\overline{\mathcal{C}}$  by construction and  $\Gamma_v \neq 0$ .



**Remark 2.1.** If  $\widetilde{w}_i > w_i$ , then  $\Gamma_{i+1} = \Gamma_i$ .

**Theorem 2.2.** The vector  $\Gamma_v$  defined in this way gives a maximum for the function  $\mu_v$  on its domain.

Before proving the theorem we need some lemmas.

**Lemma 2.3.** Let  $v = (v_1, \dots, v_{t+1}) \in \mathbb{R}^{t+1} - \{0\}$  verifying  $\sum_{i=0}^{t+1} v_i b^i = 0$ . Let  $\Gamma$  be the point in  $\overline{\mathcal{C}}$  which is closest to  $v$ . Then  $\Gamma$  achieves the maximum of  $\mu_v$ .

**Proof.** For any  $\alpha \in \mathbb{R}^{>0}$ , the vector  $\alpha\Gamma$  is also in  $\overline{\mathcal{C}}$ , so in particular  $\Gamma$  is the closest point in the line  $\alpha\Gamma$  to  $v$ . This point is the orthogonal projection of  $v$  into the line  $\alpha\Gamma$ , and the distance is

$$(2.1) \quad \|v\| \sin \beta(v, \Gamma)$$

where  $\beta(\Gamma, v)$  is the angle between  $\Gamma$  and  $v$ . But a vector  $\Gamma \in \overline{\mathcal{C}}$  minimizes (2.1) if and only if it maximizes

$$\|v\| \cos \beta(\Gamma, v) = \frac{(\Gamma, v)}{\|\Gamma\|}$$

so the lemma is proved. ■

We say that an affine hyperplane in  $\mathbb{R}^{t+1}$  separates a point  $v$  from  $\mathcal{C}$  if  $v$  is on one side of the hyperplane and all the points of  $\mathcal{C}$  are on the other side of the hyperplane.

**Lemma 2.4.** Let  $v \notin \overline{\mathcal{C}}$ . A point  $\Gamma \in \overline{\mathcal{C}} - \{0\}$  gives minimum distance to  $v$  if and only if the hyperplane  $\Gamma + (v - \Gamma)^\perp$  separates  $v$  from  $\mathcal{C}$ .

**Proof.**  $\Rightarrow$ ) Let  $\Gamma \in \overline{\mathcal{C}}$  and assume there is a point  $w \in \mathcal{C}$  on the same side of the hyperplane as  $v$ . The segment going from  $\Gamma$  to  $w$  is in  $\overline{\mathcal{C}}$  (by convexity of  $\overline{\mathcal{C}}$ ), but there are points in this segment (near  $\Gamma$ ), which are closer to  $v$  than  $\Gamma$ .

$\Leftarrow$ ) Let  $\Gamma$  be a point in  $\overline{\mathcal{C}}$  such that  $\Gamma + (v - \Gamma)^\perp$  separates  $v$  from  $\mathcal{C}$ . Let  $w \in \overline{\mathcal{C}}$  be another point. Let  $w'$  be the intersection of the hyperplane and the segment which goes from  $w$  to  $v$ . Since the hyperplane separates  $\mathcal{C}$  from  $v$ , either  $w' = w$  or  $w'$  is in the interior of the segment. Therefore

$$d(w, v) \geq d(w', v) \geq d(\Gamma, v)$$

where the last inequality follows from the fact that  $\Gamma$  is the orthogonal projection of  $v$  to the hyperplane. ■

We thank F. Presas for suggesting this lemma, which is the key to prove Theorem 2.2.

**Proof of the theorem.** Let  $\Gamma_v = (\Gamma_1, \dots, \Gamma_{t+1})$  be the vector in the hypothesis of the theorem. If  $v \in \overline{\mathcal{C}}$ , then  $\Gamma_v = v$ , and use Lemma 2.3 to conclude. If  $v \notin \overline{\mathcal{C}}$ , by Lemmas 2.3 and 2.4, it is enough to check that the hyperplane  $\Gamma_v + (v - \Gamma_v)^\perp$  separates  $v$  from  $\mathcal{C}$ .

Let  $\Gamma_v + \epsilon \in \mathcal{C}$ . The condition that  $\Gamma_v + \epsilon$  belongs to  $\mathcal{C}$  means that

$$(2.2) \quad \epsilon_i - \epsilon_{i+1} < \Gamma_{i+1} - \Gamma_i$$

The hyperplane separates  $v$  from  $\mathcal{C}$  if and only if  $(v - \Gamma_v, \epsilon) < 0$  for all such  $\epsilon$ . Therefore we calculate (using the convention  $\widetilde{w}_0 = 0$ ,  $w_0 = 0$ , and  $\widetilde{w}_{t+1} = w_{t+1} = 0$ )

$$\begin{aligned} (v - \Gamma_v, \epsilon) &= \sum_{i=1}^{t+1} b^i (v_i - \Gamma_i) \epsilon_i = \sum_{i=1}^{t+1} (-w^i + (\widetilde{w}_i - \widetilde{w}_{i-1})) \epsilon_i = \\ &= \sum_{i=1}^{t+1} ((\widetilde{w}_i - \widetilde{w}_{i-1}) - (w_i - w_{i-1})) \epsilon_i = \sum_{i=1}^{t+1} (\widetilde{w}_i - w_i) (\epsilon_i - \epsilon_{i+1}) . \end{aligned}$$

If  $\widetilde{w}_i = w_i$ , then the corresponding summand is zero. On the other hand, if  $\widetilde{w}_i > w_i$ , then  $\Gamma_{i+1} = \Gamma_i$  (Remark 2.1), and (2.2) implies  $\epsilon_i - \epsilon_{i+1} < 0$ . In any case, the summands are always non-positive, and there is at least one which is negative (because  $v \notin \overline{\mathcal{C}}$  and then  $v \neq \Gamma_v$  and  $\widetilde{w}_i > w_i$  for at least one  $i$ ). Hence

$$(v - \Gamma_v, \epsilon) < 0 .$$

■

Therefore, the function  $\mu_v(\Gamma)$  achieves its maximum for the value  $\Gamma_v \in \overline{\mathcal{C}} - \{0\}$  (or any other point on the ray  $\alpha\Gamma_v$ ) defined as the convex envelope of the graph associated to  $v$ .

### 3. PROPERTIES OF THE KEMPF FILTRATION

Let  $E$  be an unstable torsion-free sheaf over  $X$  of Hilbert polynomial  $P$ . Let  $m$  be an integer,  $m \geq m_0$  and let  $V$  be a vector space of dimension  $P(m) = h^0(E(m))$  (recall that  $m_0$  was defined after Theorem 1.3). We fix an isomorphism  $V \simeq H^0(E(m))$  and let  $V_1 \subset \dots \subset V_{t+1} = V$  be the filtration of vector spaces given by Theorem 1.4. Recall that it is called the *Kempf filtration of  $V$* . For each index  $i$ , let  $E_i^m \subset E$  the subsheaf generated by  $V_i$  under the evaluation map. We call this filtration

$$0 = E_0^m \subseteq E_1^m \subseteq E_2^m \subseteq \dots \subseteq E_t^m \subseteq E_{t+1}^m = E ,$$

the  *$m$ -Kempf filtration of  $E$* .

**Definition 3.1.** Let  $m \geq m_0$ . Given  $0 = V_0 \subset V_1 \subset \dots \subset V_{t+1} = V$  a filtration of vector spaces of  $V$ . Let

$$\begin{aligned} v_{m,i} &= m^{n+1} \cdot \frac{1}{\dim V^i \dim V} [r^i \dim V - r \dim V^i] , \\ b_m^i &= \frac{1}{m^n} \dim V^i > 0 , \\ w_m^i &= -b_m^i \cdot v_{m,i} = m \cdot \frac{1}{\dim V} [r \dim V^i - r^i \dim V] . \end{aligned}$$



Also let

$$b_{m,i} = b_m^1 + \dots + b_m^i = \frac{1}{m^n} \dim V_i ,$$

$$w_{m,i} = w_m^1 + \dots + w_m^i = m \cdot \frac{1}{\dim V} [r \dim V_i - r_i \dim V] .$$

We call the graph defined by points  $(b_{m,i}, w_{m,i})$  the graph associated to the filtration  $V_\bullet \subset V$ .

Now we can identify the Kempf function in Theorem 1.4

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} ,$$

with the function in Theorem 2.2 up to a factor which is a power of  $m$ , by defining  $v_{m,i}$ , the coordinates of vector  $v_m$ , and  $b_m^i$ , the eigenvalues of the scalar product, as in Definition 3.1. Note that  $-v_{m,i}$  are the slopes of the graph associated to the filtration  $V_\bullet \subset V$ . Here the coordinates  $\Gamma_i$  are the same as in the 1-PS defined by  $n_i$ . Also note that  $\sum_{i=1}^{t+1} v_{m,i} b_m^i = 0$ . Then, an easy calculation shows that

**Proposition 3.2.** *For every integer  $m$ , the following equality holds*

$$\mu(V_\bullet, n_\bullet) = m^{(-\frac{n}{2}-1)} \cdot \mu_{v_m}(\Gamma)$$

between the Kempf function on Theorem 1.4 and the function in Theorem 2.2.

In the following, we will omit the subindex  $m$  for the numbers  $v_{m,i}$ ,  $b_{m,i}$ ,  $w_{m,i}$  in the definition of the graph associated to a filtration of vector spaces, where it is clear from the context. Hence, given  $V \simeq H^0(E(m))$  we will refer to a filtration  $V_\bullet \subset V$  and a vector  $v = (v_1, \dots, v_{t+1})$  as the vector of the graph associated to the filtration.

**Remark 3.3.** *We introduce the factor  $m^{n+1}$  in Definition 3.1 for convenience, so that  $v_{m,i}$  and  $b_m^i$  have order zero on  $m$ , because  $\dim V = P(m)$  appears in their expressions. Then, the size of the graph does not change when  $m$  grows.*

**Lemma 3.4.** *Let  $V_1 \subset \dots \subset V_{t+1} = V$  be the Kempf filtration of  $V$  (cf. Theorem 1.4). Let  $v = (v_1, \dots, v_{t+1})$  be the vector of the graph associated to this filtration by Definition 3.1. Then*

$$v_1 < v_2 < \dots < v_t < v_{t+1} ,$$

*i.e., the graph is convex.*

**Proof.** By Theorem 1.4 the maximum of  $\mu$  among all filtrations  $V_\bullet \subset V$  and weights  $n_i > 0, \forall i$  is achieved by a unique weighted filtration  $(V_\bullet, n_\bullet)$ ,  $n_i > 0, \forall i$ , up to rescaling. Let  $V_\bullet \subset V$  be this filtration, and allow  $n_i$  to vary. By Proposition 3.2  $\mu$  is equal to  $\mu_v$  up to a constant factor. By Theorem 2.2,  $\mu_v$  achieves the maximum on  $\Gamma_v$ . The vector  $\Gamma_v$  corresponds to the weights  $n_i$  given by Theorem 1.4. Summing up, if  $V_\bullet \subset V$  is Kempf filtration of  $V$ , then the vector  $\Gamma_v = (\Gamma_1, \dots, \Gamma_{t+1})$  has  $\Gamma_i < \Gamma_{i+1}, \forall i$ .

Assume that, for the Kempf filtration of  $V$ , there exists some  $i$  such that  $v_i \geq v_{i+1}$ . Then  $v \notin \mathcal{C}$  and, by Lemma 2.3,  $\Gamma_v \in \overline{\mathcal{C}} \setminus \mathcal{C}$ , which means that there exists some  $j$  with  $\Gamma_j = \Gamma_{j+1}$ , but we have just seen that this is impossible. ■

**Lemma 3.5.** *Let  $V_1 \subset \dots \subset V_{t+1} = V$  be the Kempf filtration of  $V$  (cf. Theorem 1.4). Let  $W$  be a vector space with  $V_i \subset W \subset V_{i+1}$  and consider the new filtration  $V'_\bullet \subset V$*

$$(3.1) \quad \begin{array}{ccccccccccccccc} 0 & \subset & V'_1 & \subset & \dots & \subset & V'_i & \subset & V'_{i+1} & \subset & V'_{i+2} & \subset & \dots & \subset & V'_{t+2} = V \\ \parallel & & \parallel & & & & \parallel & & \parallel & & \parallel & & & & \parallel \\ 0 & \subset & V_1 & & & & V_i & \subset & W & \subset & V_{i+1} & \subset & \dots & \subset & V_{t+1} = V \end{array}$$

*Then,  $v'_{i+1} \geq v_{i+1}$ . We say that the Kempf filtration is the convex envelope of every refinement.*

**Proof.** The graph associated to  $V'_\bullet \subset V$  has one more point than the graph associated to  $V_\bullet \subset V$ , hence it is a refinement of the graph associated to Kempf filtration of  $V$ . Therefore the convex envelope of the graph associated to  $v'$  has to be equal to the graph associated to  $v$ , and this happens only when the extra point associated to  $W$  is not above the graph associated to  $v$ , which means that the slope  $-v'_{i+1}$  has to be less or equal than  $-v_{i+1}$ . ■

Later on, we will check that, for  $m$  large enough, the  $m$ -Kempf filtration stabilizes in the sense  $E_i^m = E_i^{m+l}$ ,  $\forall i, \forall l > 0$ . The  $m$ -Kempf filtration for  $m \gg 0$  will be called the Kempf filtration of  $E$ , and the goal of this article is to show that it coincides with the Harder-Narasimhan filtration of  $E$ .

**Lemma 3.6** (Simpson). [Si, Corollary 1.7] or [HL1, Lemma 2.2] *Let  $r > 0$  be an integer. Then there exists a constant  $B$  with the following property: for every torsion free sheaf  $E$  with  $0 < \text{rk}(E) \leq r$ , we have*

$$h^0(E) \leq \frac{1}{g^{n-1}n!} ((\text{rk}(E) - 1)([\mu_{\max}(E) + B]_+)^n + ([\mu_{\min}(E) + B]_+)^n),$$

where  $g = \deg \mathcal{O}_X(1)$ ,  $[x]_+ = \max\{0, x\}$ , and  $\mu_{\max}(E)$  (respectively  $\mu_{\min}(E)$ ) is the maximum (resp. minimum) slope of the Mumford-semistable factors of the Harder-Narasimhan filtration of  $E$ .

**Remark 3.7.** *Recall that the Harder-Narasimhan filtration with Gieseker stability is a refinement of the one with Mumford stability, with the inequalities holding between polynomials of their leading coefficients.*

We denote

$$P_{\mathcal{O}_X}(m) = \frac{\alpha_n}{n!} m^n + \frac{\alpha_{n-1}}{(n-1)!} m^{n-1} + \dots + \frac{\alpha_1}{1!} m + \frac{\alpha_0}{0!}$$

the Hilbert polynomial of  $\mathcal{O}_X$ , then  $\alpha_n = g$ . Let

$$P(m) = \frac{rg}{n!} m^n + \frac{d + r\alpha_{n-1}}{(n-1)!} m^{n-1} + \dots$$

the Hilbert polynomial of the sheaf  $E$ , where  $d$  is the degree and  $r$  is the rank. Let us call  $A = d + r\alpha_{n-1}$ , so

$$P(m) = \frac{rg}{n!} m^n + \frac{A}{(n-1)!} m^{n-1} + \dots$$

Let us define

$$(3.2) \quad C = \max\{r|\mu_{\max}(E)| + \frac{d}{r} + r|B| + |A| + 1, 1\},$$

a positive constant.

**Proposition 3.8.** *Given an integer  $m$  and a vector space  $V \simeq H^0(E(m))$ , we have the Kempf filtration  $V_\bullet \subset V \simeq H^0(E(m))$  and, by evaluation, the  $m$ -Kempf filtration  $E_\bullet^m \subset E$ . There exists an integer  $m_2$  such that for  $m \geq m_2$ , each filter in the  $m$ -Kempf filtration of  $E$  has slope  $\mu(E_i^m) \geq \frac{d}{r} - C$ .*

**Proof.** Choose an  $m_1 \geq m_0$  such that for  $m \geq m_1$

$$[\mu_{\max}(E) + gm + B]_+ = \mu_{\max}(E) + gm + B$$

and

$$[\frac{d}{r} - C + gm + B]_+ = \frac{d}{r} - C + gm + B.$$

Now let  $m \geq m_1$  and let

$$0 \subseteq E_1^m \subseteq E_2^m \subseteq \dots \subseteq E_t^m \subseteq E_{t+1}^m = E$$

be the  $m$ -Kempf filtration.

Suppose we have a filter  $E_i^m \subseteq E$ , of rank  $r_i$  and degree  $d_i$ , such that  $\mu(E_i^m) < \frac{d}{r} - C$ . The subsheaf  $E_i^m(m) \subset E(m)$  satisfies the estimate in Lemma 3.6,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!} ((r_i - 1)([\mu_{\max}(E_i^m) + gm + B]_+)^n + ([\mu_{\min}(E_i^m) + gm + B]_+)^n),$$

where  $\mu_{\max}(E_i^m(m)) = \mu_{\max}(E_i^m) + gm$  and similarly for  $\mu_{\min}$ .

Note that  $\mu_{\max}(E_i^m) \leq \mu_{\max}(E)$  and  $\mu_{\min}(E_i^m) \leq \mu(E_i^m) < \frac{d}{r} - C$ , so

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!} ((r_i - 1)([\mu_{\max}(E) + gm + B]_+)^n + ([\frac{d}{r} - C + gm + B]_+)^n),$$

and, by choice of  $m$ ,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!} ((r_i - 1)(\mu_{\max}(E) + gm + B)^n + (\frac{d}{r} - C + gm + B)^n) = G(m),$$

where

$$G(m) = \frac{1}{g^{n-1}n!} [r_i g^n m^n + n g^{n-1} ((r_i - 1)\mu_{\max}(E) + \frac{d}{r} - C + r_i B) m^{n-1} + \dots].$$

Recall that, by Definition 3.1, to such filtration we associate a graph with heights, for each  $j$ ,

$$w_j = w^1 + \dots + w^j = m \cdot \frac{1}{\dim V} [r \dim V_j - r_j \dim V].$$

To reach a contradiction, it is enough to show that  $w_i < 0$ . In that case, the graph has to be convex by Lemma 3.4. If  $w_i < 0$  there is a  $j < i$  such that  $-v_j < 0$ , because the graph starts on the origin. Hence, the rest of the slopes of the graph are negative,  $-v_k < 0$ ,  $k \geq i$ , because the slopes have to be decreasing. Then  $w_i > w_{i+1} > \dots w_{t+1}$ , and  $w_{t+1} < 0$ . But it is

$$w_{t+1} = m \cdot \frac{1}{\dim V} [r \dim V_{t+1} - r_{t+1} \dim V] = 0,$$

because  $r_{t+1} = r$  and  $V_{t+1} = V$ , then the contradiction.

Let us show that  $w_i < 0$ . Since  $E_i^m(m)$  is generated by  $V_i$  under the evaluation map, it is  $\dim V_i \leq h^0(E_i^m(m))$ , hence

$$w_i = \frac{m}{\dim V} [r \dim V_i - r_i \dim V] \leq$$

$$\leq \frac{m}{P(m)} [rh^0(E_i^m(m)) - r_i P(m)] \leq \frac{m}{P(m)} [rG(m) - r_i P(m)] .$$

Hence,  $w_i < 0$  is equivalent to

$$\Psi(m) = rG(m) - r_i P(m) < 0 ,$$

where  $\Psi(m) = \xi_n m^n + \xi_{n-1} m^{n-1} + \dots + \xi_1 m + \xi_0$  is an  $n$ -order polynomial. Let us calculate the  $n$ -coefficient:

$$\xi_n = (rG(m) - r_i P(m))_n = r \frac{r_i g}{n!} - r_i \frac{r g}{n!} = 0 .$$

Then,  $\Psi(m)$  has no coefficient in order  $n$ . Let us calculate the  $(n-1)$ -coefficient:

$$\xi_{n-1} = (rG(m) - r_i P(m))_{n-1} = (rG_{n-1} - r_i \frac{A}{(n-1)!})$$

where  $G_{n-1}$  is the  $(n-1)$ -coefficient of the polynomial  $G(m)$ ,

$$\begin{aligned} G_{n-1} &= \frac{1}{g^{n-1} n!} n g^{n-1} ((r_i - 1) \mu_{\max}(E) + \frac{d}{r} - C + r_i B) = \\ &\frac{1}{(n-1)!} ((r_i - 1) \mu_{\max}(E) + \frac{d}{r} - C + r_i B) \leq \\ &\frac{1}{(n-1)!} ((r_i - 1) |\mu_{\max}(E)| + \frac{d}{r} - C + r_i |B|) \leq \\ &\frac{1}{(n-1)!} (r |\mu_{\max}(E)| + \frac{d}{r} - C + r |B|) < \frac{-|A|}{(n-1)!} , \end{aligned}$$

last inequality coming from the definition of  $C$  in (3.2). Then

$$\xi_{n-1} < r \left( \frac{-|A|}{(n-1)!} \right) - r_i \frac{A}{(n-1)!} = \frac{-r|A| - r_i A}{(n-1)!} < 0$$

because  $-r|A| - r_i A < 0$ .

Therefore  $\Psi(m) = \xi_{n-1} m^{n-1} + \dots + \xi_1 m + \xi_0$  with  $\xi_{n-1} < 0$ , so there exists  $m_2 \geq m_1$  such that for  $m \geq m_2$  we will have  $\Psi(m) < 0$  and  $w_i < 0$ , then the contradiction. ■

**Proposition 3.9.** *There exists an integer  $m_3$  such that for  $m \geq m_3$  the sheaves  $E_i^m$  and  $E^{m,i} = E_i^m / E_{i-1}^m$  are  $m_3$ -regular. In particular their higher cohomology groups vanish and they are generated by global sections.*

**Proof.** Note that  $\mu(E_i^m) \leq \mu_{\max}(E)$ . Then, although  $E_i^m$  depends on  $m$ , its slope is bounded above and below by numbers which do not depend on  $m$ , (cf. Proposition 3.8) and furthermore it is a subsheaf of  $E$ . Hence, the set of possible isomorphism classes for  $E_i^m$  is bounded. Apply Serre vanishing theorem choosing  $m_3 \geq m_2$ . ■

**Proposition 3.10.** *Let  $m \geq m_3$ . For each filter  $E_i^m$  in the  $m$ -Kempf filtration, we have  $\dim V_i = h^0(E_i^m(m))$ , therefore  $V_i \cong H^0(E_i^m(m))$ .*

**Proof.** Let  $V_\bullet \subseteq V$  be the Kempf filtration of  $V$  (cf. Theorem 1.4) and let  $E_\bullet^m \subseteq E$  be the  $m$ -Kempf filtration of  $E$ . We know that each  $V_i$  generates the subsheaf  $E_i^m$ , by definition, then we have the following diagram:

$$\begin{array}{ccccccc}
0 & \subset & V_1 & \subset & V_2 & \subset & \cdots \subset V_{t+1} = V \\
& & \cap & & \cap & & \parallel \\
& & H^0(E_1^m(m)) & \subset & H^0(E_2^m(m)) & \subset & \cdots \subset H^0(E_{t+1}^m(m)) = H^0(E(m))
\end{array}$$

Suppose there exists an index  $i$  such that  $V_i \neq H^0(E_i^m(m))$ . Let  $i$  be the index such that  $V_i \neq H^0(E_i^m(m))$  and  $\forall j > i$  it is  $V_j = H^0(E_j^m(m))$ . Then we have the diagram:

$$\begin{array}{ccc}
V_i & \subset & V_{i+1} \\
\cap & & \parallel \\
H^0(E_i^m(m)) & \subseteq & H^0(E_{i+1}^m(m))
\end{array}
\quad (3.3)$$

Therefore  $V_i \subsetneq H^0(E_i^m(m)) \subseteq V_{i+1}$  and we can consider a new filtration by adding the filter  $H^0(E_i^m(m))$ :

$$\begin{array}{ccccc}
V_i & \subset & H^0(E_i^m(m)) & \subset & V_{i+1} \\
\parallel & & \parallel & & \parallel \\
V'_i & & V'_{i+1} & & V'_{i+2}
\end{array}
\quad (3.4)$$

Note that we are in situation of Lemma 3.5, where  $W = H^0(E_i^m(m))$ , filtration  $V_\bullet$  is (3.3) and filtration  $V'_\bullet$  is (3.4).

The graph associated to filtration  $V_\bullet$ , by Definition 3.1, is given by the points

$$(b_i, w_i) = \left( \frac{\dim V_i}{m^n}, \frac{m}{\dim V} (r \dim V_i - r_i \dim V) \right),$$

where the slopes of the graph are given by

$$\begin{aligned}
-v_i &= \frac{w_i}{b_i} = \frac{w_i - w_{i-1}}{b_i - b_{i-1}} = \\
&= \frac{m^{n+1}}{\dim V} \left( r - r^i \frac{\dim V}{\dim V^i} \right) \leq \frac{m^{n+1}}{\dim V} \cdot r := R
\end{aligned}$$

and equality holds if and only if  $r^i = 0$ .

Now, the new point which appears in graph of the filtration  $V'_\bullet$  is

$$Q = \left( \frac{h^0(E_i^m(m))}{m^n}, \frac{m}{\dim V} (r h^0(E_i^m(m)) - r_i \dim V) \right).$$

Point  $Q$  joins two new segments appearing in this new graph. The slope of the segment between  $(b_i, w_i)$  and  $Q$  is, by a similar calculation,

$$-v'_{i+1} = \frac{m^{n+1}}{\dim V} \cdot r = R.$$

By Lemma 3.4, the graph is convex, so  $v_1 < v_2 < \dots < v_{t+1}$ . As  $E_1^m$  is a non-zero torsion-free sheaf, it has positive rank  $r_1 = r^1$  and so it follows  $v_1 > -R$ . On the other hand, by Lemma 3.5,  $v'_{i+1} \geq v_{i+1}$ . Hence

$$-R < v_1 < v_2 < \dots < v_{i+1} \leq v'_{i+1} = -R,$$

which is a contradiction.

Therefore,  $\dim V_i = h^0(E_i^m(m))$ , for every filter in the  $m$ -Kempf filtration.  $\blacksquare$

**Corollary 3.11.** *For every filter  $E_i^m$  in the  $m$ -Kempf filtration, it is  $r^i > 0$ .*

**Proof.** We have seen that  $r^i = 0$  is equivalent to  $-v_i = R$ . Then the result follows from Proposition 3.10 because it is  $r^1 = r_1 > 0$  and  $-R < v_1 < v_2 < \dots < v_{t+1}$ . ■

#### 4. THE $m$ -KEMPF FILTRATION STABILIZES

In Proposition 3.9 we have seen that, for any  $m \geq m_3$ , all the filters  $E_i^m$  of the  $m$ -Kempf filtration of  $E$  are  $m_3$ -regular. Hence,  $E_i^m(m_3)$  is generated by the subspace  $H^0(E_i^m(m_3))$  of  $H^0(E(m_3))$ , and the filtration of sheaves

$$E_1^m \subset E_2^m \subset \dots \subset E_{t_m}^m \subset E_{t_m+1}^m = E$$

is the filtration associated to the filtration of vector spaces

$$H^0(E_1^m(m_3)) \subset H^0(E_2^m(m_3)) \subset \dots \subset H^0(E_{t_m}^m(m_3)) \subset H^0(E_{t_m+1}^m(m_3)) = H^0(E(m_3))$$

by the evaluation map. Note that the dimension of the vector space  $H^0(E(m_3))$  does not depend on  $m$  and, by Corollary 3.11, the length  $t_m + 1$  of the  $m$ -Kempf filtration of  $E$  is at most equal to  $r$ , the rank of  $E$ , a bound which does not also depend on  $m$ .

We call  $m$ -type to the tuple of different Hilbert polynomials appearing in the  $m$ -Kempf filtration of  $E$

$$(P_1^m, \dots, P_{t_m+1}^m),$$

where  $P_i^m := P_{E_i^m}$ . Note that  $P^{i,m} := P_{E_i^m/E_{i-1}^m} = P_{E_i^m} - P_{E_{i-1}^m}$ , so they are defined in terms of elements of each  $m$ -type.

**Proposition 4.1.** *For all integers  $m \geq m_3$ , the set of possible  $m$ -types*

$$\mathcal{P} = \{(P_1^m, \dots, P_{t_m+1}^m)\}$$

*is finite.*

**Proof.** Once we fix  $V \cong H^0(E(m_3))$  of dimension  $h^0(E(m_3))$  (which does not depend on  $m$ ), all the possible filtrations by vector subspaces are parametrized by a finite-type scheme. Therefore the set of all possible  $m$ -Kempf filtrations of  $E$ , for  $m \geq m_3$ , is bounded and  $\mathcal{P}$  is finite. ■

Recall that the vector  $v$  can be recovered from the filtration  $V_\bullet \subset V$  and the vector  $\Gamma$  from the weights  $n_i$ . Then, given  $m$ , the  $m$ -Kempf filtration achieves the maximum for the function  $\mu(V_\bullet, n_\bullet)$ , which is the same, by Proposition 3.2, as achieving the maximum for the function

$$\mu_v(\Gamma) = \frac{(\Gamma, v)}{\|\Gamma\|},$$

among all filtrations  $V_\bullet \subset V$  and vectors  $\Gamma \in \mathcal{C} - \{0\}$ , where

$$\mathcal{C} = \{x \in \mathbb{R}^{t+1} : x_1 < x_2 < \dots < x_{t+1}\}.$$

By Definition 3.1 we associate a graph to the  $m$ -Kempf filtration, given by  $v_m$ . Recall that, by Lemma 3.4 the graph is convex, meaning  $v_m \in \mathcal{C}$ , which implies  $\Gamma_{v_m} = v_m$  by Lemma 2.3. Then, given  $v_m$  associated to the  $m$ -Kempf filtration

$$(4.1) \quad \max_{\Gamma \in \mathcal{C}} \mu_{v_m}(\Gamma) = \mu_{v_m}(\Gamma_{v_m}) = \frac{(\Gamma_{v_m}, v_m)}{\|\Gamma_{v_m}\|} = \frac{(v_m, v_m)}{\|v_m\|} = \|v_m\|,$$

where recall that we defined in 3.1

$$v_{m,i} = m^{n+1} \cdot \frac{1}{\dim V^i \dim V} [r^i \dim V - r \dim V^i]$$

$$b_m^i = \frac{1}{m^n} \cdot \dim V^i ,$$

and thanks to Propositions 3.9 and 3.10, we can rewrite

$$v_{m,i} = m^{n+1} \cdot \frac{1}{P^{i,m}(m)P(m)} [r^i P(m) - r P^{i,m}(m)]$$

$$b_m^i = \frac{1}{m^n} \cdot P^{i,m}(m) .$$

Let

$$v_{m,i}(l) = m^{n+1} \cdot \frac{1}{P^{i,m}(l)P(l)} [r^i P(l) - r P^{i,m}(l)]$$

and let us define

$$\Theta_m(l) = (\mu_{v_m(l)}(\Gamma_{v_m(l)}))^2 = \|v_m(l)\|^2 ,$$

where the second equality follows by an argument similar to (4.1). Note that  $\Theta_m(l)$  is a rational function on  $l$ . Let

$$\mathcal{A} = \{\Theta_m : m \geq m_3\}$$

which is a finite set by Proposition 4.1. We say that  $f_1 \prec f_2$  for two rational functions, if the inequality  $f_1(l) < f_2(l)$  holds for  $l \gg 0$ , and let  $K$  be the maximal function in the finite set  $\mathcal{A}$ , with respect to the defined ordering.

Note that the value  $\Theta_m(m)$  is the square of the maximum of Kempf's function  $\mu_{v_m}(\Gamma)$ , by (4.1), achieved for the maximal filtration  $V_\bullet \subset V \simeq H^0(E(m))$  of vector spaces which gives the vector  $v_m$ . This weighted filtration is the only one which gives the value  $\sqrt{\Theta_m(m)}$  for the Kempf function.

**Lemma 4.2.** *There exists an integer  $m_4 \geq m_3$  such that  $\forall m \geq m_4$ ,  $\Theta_m = K$ .*

**Proof.** Choose  $m_4$  such that  $K(l) \geq \Theta_m(l)$ ,  $\forall l \geq m_4$  and every  $\Theta_m \in \mathcal{A}$  with equality only when  $\Theta_m = K$ , and let  $m \geq m_4$ . Given that the Kempf function achieves the maximum over all possible filtrations and weights (c.f. Theorem 1.4), we have  $\Theta_m(m) \geq K(m)$ , because  $K$  is another rational function built with other  $m'$ -type, i.e., other values for the polynomials appearing on the rational function. Combining both inequalities we obtain  $\Theta_m(m) = K(m)$  for all  $m \geq m_4$ . ■

**Proposition 4.3.** *Let  $l_1$  and  $l_2$  be integers with  $l_1 \geq l_2 \geq m_4$ . Then the  $l_1$ -Kempf filtration of  $E$  is equal to the  $l_2$ -Kempf filtration of  $E$ .*

**Proof.** By construction, the filtration

$$(4.2) \quad H^0(E_1^{l_1}(l_1)) \subset H^0(E_2^{l_1}(l_1)) \subset \cdots \subset H^0(E_{t_1}^{l_1}(l_1)) \subset H^0(E_{t_1+1}^{l_1}(l_1)) = H^0(E(l_1))$$

is the  $l_1$ -Kempf filtration of  $V \simeq H^0(E(l_1))$ . Now consider the filtration  $V'_\bullet \subset V \simeq H^0(E(l_1))$  defined as follows

$$(4.3) \quad H^0(E_1^{l_2}(l_1)) \subset H^0(E_2^{l_2}(l_1)) \subset \cdots \subset H^0(E_{t_2}^{l_2}(l_1)) \subset H^0(E_{t_2+1}^{l_2}(l_1)) = H^0(E(l_1)) .$$

We have to prove that (4.3) is in fact the  $l_1$ -Kempf filtration of  $V \simeq H^0(E(l_1))$ .

Since  $l_1, l_2 \geq m_4$ , by Lemma 4.2 we have  $\Theta_{l_1} = \Theta_{l_2} = K$ . Then,  $\Theta_{l_1}(l_1) = \Theta_{l_2}(l_1)$  and, by uniqueness of the Kempf filtration (c.f. Theorem 1.4), the filtrations (4.2) and (4.3) coincide. Since, in particular  $l_1, l_2 \geq m_3$ ,  $E_i^{l_1}$  and  $E_i^{l_2}$  are  $l_1$ -regular by Proposition 3.9. Hence,  $E_i^{l_1}(l_1)$  and  $E_i^{l_2}(l_1)$  are generated by their global sections (c.f. Lemma 1.1)  $H^0(E_i^{l_1}(l_1)) = H^0(E_i^{l_2}(l_1))$ , which are equal by the previous argument, therefore  $E_i^{l_1}(l_1) = E_i^{l_2}(l_1)$ . By tensoring with  $\mathcal{O}_X(-l_1)$ , this implies that the filtrations  $E_{\bullet}^{l_1} \subset E$  and  $E_{\bullet}^{l_2} \subset E$  coincide. ■

**Definition 4.4.** *If  $m \geq m_4$ , the  $m$ -Kempf filtration of  $E$  is called the Kempf filtration of  $E$ ,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E.$$

## 5. KEMPF FILTRATION IS HARDER-NARASIMHAN FILTRATION

Recall that the Kempf theorem (c.f. Theorem 1.4) asserts that given an integer  $m$  and  $V \simeq H^0(E(m))$ , there exists a unique weighted filtration of vector spaces  $V_{\bullet} \subseteq V$  which gives maximum for the Kempf function

$$\mu(V_{\bullet}, n_{\bullet}) = \frac{\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V} (r^i \dim V - r \dim V^i)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}}.$$

This filtration induces a filtration of sheaves, called the Kempf filtration of  $E$ ,

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

which is independent of  $m$ , for  $m \geq m_4$ , by Proposition 4.3, hence it only depends on  $E$ . From now on, we assume  $m \geq m_4$ .

In the previous sections, based on the fact we can rewrite the Kempf function as a certain scalar product divided by a norm (c.f. Proposition 3.2), we saw that Kempf filtration is encoded by a convex graph (c.f. Lemma 3.4). We can express the data related to the filtration of vector spaces to the data of filtration of sheaves. Since  $m \geq m_3$ , the sheaves  $E_i$  and  $E^i$  are  $m$ -regular  $\forall i$ , and

$$(5.1) \quad \begin{aligned} \dim V_i &= h^0(E_i(m)) = P_{E_i}(m) =: P_i(m) \\ \dim V^i &= h^0(E^i(m)) = P_{E^i}(m) =: P^i(m) \end{aligned}$$

(c.f. Proposition 3.9 and Proposition 3.10). Recall that the Kempf function is a rational function on  $m$ , with order  $m^{-\frac{n}{2}-1}$  at zero (c.f. Proposition 3.2) then we consider the function  $\mu$ , where

$$\mu = m^{\frac{n}{2}+1} \cdot \mu(V_{\bullet}, m_{\bullet}) = \mu_{v_m}(\Gamma)$$

and, making the substitutions (5.1), and using the relation  $\gamma_i = \frac{r}{P} \Gamma_i$ ,

$$\mu = m^{\frac{n}{2}+1} \cdot \frac{\sum_{i=1}^{t+1} \frac{\gamma_i}{r} [(r^i P - r P^i)]}{\sqrt{\sum_{i=1}^{t+1} P^i \frac{P^2}{r^2} \gamma_i^2}},$$

which we see as a rational function on  $m$  (since  $P$  and  $P^i$  are polynomials on  $m$ ). Therefore we get

$$\mu = m^{\frac{n}{2}+1} \cdot \frac{1}{P} \frac{\sum_{i=1}^{t+1} \gamma_i [r^i P - r P^i]}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}}.$$



**Proposition 5.1.** *Given a sheaf  $E$ , there exists a unique filtration*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

*with positive weights  $n_1, \dots, n_t$ ,  $n_i = \frac{\gamma_{i+1} - \gamma_i}{r}$ , which gives maximum for the rational function*

$$\mu = m^{\frac{n}{2}+1} \cdot \frac{\sum_{i=1}^{t+1} P^i \gamma_i [\frac{r^i}{P^i} - \frac{r}{P}]}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}}.$$

Similarly, we had defined the coordinates  $v_i$  (slopes of segments of the graph), as

$$v_i = m^{n+1} \cdot \left[ \frac{r^i}{P^i} - \frac{r}{P} \right]$$

Therefore we can express the function  $\mu$  as

$$\mu = m^{-\frac{n}{2}} \cdot \frac{\sum_{i=1}^{t+1} P^i \gamma_i v_i}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}} = m^{-\frac{n}{2}} \cdot \frac{(\gamma, v)}{\|\gamma\|},$$

where the scalar product is given by the diagonal matrix

$$\begin{pmatrix} P^1 & & & 0 \\ & P^2 & & \\ & & \ddots & \\ 0 & & & P^{t+1} \end{pmatrix}$$

**Proposition 5.2.** *Given the Kempf filtration of a sheaf  $E$ ,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

*it verifies*

$$\frac{P^1}{r^1} > \frac{P^2}{r^2} > \cdots > \frac{P^{t+1}}{r^{t+1}}$$

**Proof.** The coordinates of the vector  $v$  associated to the filtration are, for  $m$  large enough,  $v_i = m^{n+1} \cdot (\frac{r^i}{P^i} - \frac{r}{P})$ . Now apply Lemma 3.4 which say that  $v$  is convex, i.e.  $v_1 < \dots < v_{t+1}$ . ■

**Proposition 5.3.** *Given the Kempf filtration of a sheaf  $E$ ,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E,$$

*each one of the blocks  $E^i = E_i/E_{i-1}$  is semistable.*

**Proof.** Consider the graph associated to the Kempf filtration of  $E$ . Suppose that any of the blocks has a destabilizing subsheaf. Then, it corresponds to a point above of the graph of the filtration. The graph obtained by adding this new point is a refinement of the graph of the Kempf filtration, whose convex envelope is not the original graph, which contradicts Lemma 3.5. ■

**Corollary 5.4.** *The Kempf filtration of a sheaf  $E$  coincides with its Harder-Narasimhan filtration.*

**Proof.** By Propositions 5.2 and 5.3 the Kempf filtration verifies the two properties of the Harder-Narasimhan filtration. By uniqueness of the Harder-Narasimhan filtration both filtrations coincide. ■

## 6. KEMPF FILTRATION FOR HOLOMORPHIC PAIRS

Let  $X$  be a smooth complex projective variety. Let us consider pairs

$$(E, \varphi : E \rightarrow \mathcal{O}_X)$$

given by a rank  $r$  vector bundle with fixed determinant  $\det(E) \cong \Delta$  and a morphism to the trivial line bundle  $\mathcal{O}_X$ .

Let  $\delta$  be a polynomial of degree at most  $\dim X - 1$  and positive leading coefficient.

**Definition 6.1.** A pair  $(E, \varphi)$  is  $\delta$ -(semi)stable if for all weighted filtrations

$$\sum_{i=1}^t n_i (r P_{E_i} - r_i P_E) + \delta \sum_{i=1}^t n_i (r_i - \epsilon(E_i) r) (\leq) 0 ,$$

where  $\epsilon(E_i) = 1$  if  $\varphi|_{E_i} \neq 0$  and  $\epsilon(E_i) = 0$  otherwise.

**Definition 6.2.** Given a pair  $(E, \varphi : E \rightarrow \mathcal{O}_X)$ , let  $(F, \varphi|_F)$  be a subpair where  $F \subset E$  is a subsheaf and  $\varphi|_F$  is the restriction of the morphism  $\varphi$ . Let  $E' = E/F$  and we call  $(E', \varphi|_{E'})$  a quotient pair of  $(E, \varphi)$  where, if  $\varphi|_F \neq 0$ , define  $\varphi|_{E'} = 0$ , and if  $\varphi|_F = 0$ ,  $\varphi|_{E'}$  is the induced morphism in the quotient sheaf. For every pair  $(G, \varphi|_G)$ , define  $\epsilon(G) = 1$  if  $\varphi|_G \neq 0$  and  $\epsilon(G) = 0$  otherwise. We define a morphism of pairs  $(E, \varphi) \rightarrow (F, \psi)$  as morphism of sheaves  $\alpha : E \rightarrow F$  such that  $\psi \circ \alpha = \varphi$ .

**Definition 6.3.** Let  $(G, \varphi|_G)$  be a holomorphic pair. We define the corrected Hilbert polynomial of  $(G, \varphi|_G)$  as

$$\overline{P}(G) = P_G - \delta \epsilon(G)$$

Note that the exact sequence of holomorphic pairs

$$0 \rightarrow F \rightarrow E \rightarrow E' \rightarrow 0$$

verify

$$\overline{P}(E) = \overline{P}(F) + \overline{P}(E')$$

for the corrected polynomials.

From Definition 6.1 it can be directly deduced the following equivalent definition, which appears on [HL1]

**Definition 6.4.** A pair  $(E, \varphi)$  is  $\delta$ -unstable if and only if there exists a subpair  $(F, \varphi|_F)$  with  $\frac{\overline{P}(F)}{\text{rk } F} > \frac{\overline{P}(E)}{\text{rk } E}$ .

We will recall the construction of the moduli space of  $\delta$ -semistable pairs with fixed polynomial  $P$  and fixed determinant  $\det(E) \simeq \Delta$ . This was done in [HL0] following Gieseker's ideas, and in [HL1] following Simpson's ideas. Here we will use Gieseker's method (although [HL0] assumes that  $X$  is a curve or a surface, thanks to Simpson's bound [Si, Corollary 1.7], we can follow Gieseker's method for any dimension). Let  $m$  a large integer, so that  $E$  is  $m$ -regular for all semistable  $E$  (c.f. [Ma, Corollary 3.3.1 and Proposition 3.6]). Let  $V$  be a vector space of dimension  $p := P(m)$ . Given an isomorphism  $V \cong H^0(E(m))$  we obtain a quotient

$$q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E ,$$

hence a homomorphism

$$Q : \wedge^r V \cong \wedge^r H^0(E(m)) \longrightarrow H^0(\wedge^r(E(m))) \cong H^0(\Delta(rm)) =: A$$

and points

$$Q \in \text{Hom}(\wedge^r V, A) \quad \overline{Q} \in \mathbb{P}(\text{Hom}(\wedge^r V, A)) .$$

The morphism  $\varphi : E \rightarrow \mathcal{O}_X$  induces a homomorphism

$$\Phi : V = H^0(E(m)) \rightarrow H^0(\mathcal{O}_X(m)) =: B$$

and hence points

$$\Phi \in \text{Hom}(V, B) \quad \overline{\Phi} \in \mathbb{P}(\text{Hom}(V, B)) .$$

If we change the isomorphism  $V \cong H^0(E(m))$  by a homothecy, we obtain another point in the line defined by  $\Phi$ , but the point  $\overline{\Phi}$  does not change.

An argument similar to the one in (1.4) shows that the "minimal relevant weight" of the action of a 1-PS over  $\mathbb{P}(\text{Hom}(V, B))$  is

$$(6.1) \quad \mu(\overline{\Phi}, V_\bullet, n_\bullet) = \min\{\Gamma_i : \Phi|_{E_{V_i}} \neq 0\} = \min\{\Gamma_i : \varphi|_{E_{V_i}} \neq 0\}$$

where  $E_{V_i}$  is the subsheaf of  $E$  generated by  $V_i$ . If  $j$  is the index giving minimum in (6.1), we will define  $\epsilon_i(\overline{\Phi}, V_\bullet) = 1$  if  $i \geq j$  and  $\epsilon_i(\overline{\Phi}, V_\bullet) = 0$  otherwise. We will denote  $\epsilon_i(\overline{\Phi}, V_\bullet)$  by just  $\epsilon_i(\overline{\Phi})$  if the filtration  $V_\bullet$  is clear from the context. Let us call  $\epsilon^i(\overline{\Phi}) = \epsilon_i(\overline{\Phi}) - \epsilon_{i-1}(\overline{\Phi})$  and note that  $\epsilon_i(\overline{\Phi}, V_\bullet) = \epsilon(E_{V_i})$ , in Definition 6.2. Also note that the definition of  $\epsilon_i(\overline{\Phi})$  is independent of the weights  $n_\bullet$  or the vector  $\Gamma$  associated to them. Therefore,

$$(6.2) \quad \mu(\overline{\Phi}, V_\bullet, n_\bullet) = \sum_{i=1}^t n_i (\dim V_i - \epsilon_i(\overline{\Phi}) \dim V) = \sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V} (\epsilon^i(\overline{\Phi}) \dim V - \dim V^i).$$

By the Hilbert-Mumford criterion, a point

$$(\overline{Q}, \overline{\Phi}) \in \mathbb{P}(\text{Hom}(\wedge^r V, A)) \times \mathbb{P}(\text{Hom}(V, B))$$

is GIT (semi)-stable with respect to the natural linearization on  $\mathcal{O}(a_1, a_2)$  if and only if for all weighted filtrations

$$\mu(\overline{Q}, V_\bullet, n_\bullet) + \frac{a_2}{a_1} \mu(\overline{\Phi}, V_\bullet, n_\bullet) (\leq) 0$$

Using the calculations in (1.5) and (6.2) this can be stated as follows:

**Proposition 6.5 (Hilbert-Mumford).** *A point  $(\overline{Q}, \overline{\Phi})$  is GIT  $a_2/a_1$ -(semi)stable if for all weighted filtrations  $(V_\bullet, n_\bullet)$*

$$\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V) + \frac{a_2}{a_1} \sum_{i=1}^t n_i (\dim V_i - \epsilon_i(\overline{\Phi}) \dim V) (\leq) 0$$

**Theorem 6.6.** *Let  $(E, \varphi)$  be a pair. There exists an integer  $m_0$  such that, for  $m \geq m_0$ , the associated point  $(\overline{Q}, \overline{\Phi})$  is GIT  $a_2/a_1$ -(semi)stable if and only if the pair is  $\delta$ -(semi)stable, where*

$$\frac{a_2}{a_1} = \frac{r\delta(m)}{P_E(m) - \delta(m)}$$

Let  $(E, \varphi)$  be an unstable pair. Let  $m \geq m_0$  an integer and let  $V$  be a vector space of dimension  $P(m) = h^0(E(m))$  ( $m_0$  now defined in the construction of the moduli space of pairs).

Given a filtration of vector subspaces  $V_1 \subset \cdots \subset V_{t+1} = V$  and rational numbers  $n_1, \dots, n_t > 0$ , i.e., given a weighted filtration, we define now the function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V) + \frac{a_2}{a_1} \sum_{i=1}^t n_i (\dim V_i - \epsilon_i(\overline{\Phi}) \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V_i \Gamma_i^2}},$$

which we call *Kempf function*, as in the case of sheaves.

Fix an isomorphism  $V \simeq H^0(E(m))$ . Let

$$(6.3) \quad 0 = V_0 \subset V_1 \subset \cdots \subset V_{t+1} = V$$

the *Kempf filtration* of vector spaces given by Theorem 1.4, and let

$$(6.4) \quad 0 = E_0^m \subsetneq E_1^m \subsetneq E_2^m \subsetneq \cdots \subsetneq E_t^m \subsetneq E_{t+1}^m = E,$$

the *m-Kempf filtration of the pair*  $(E, \varphi)$ , where  $E_i^m \subset E$  is the subsheaf generated by  $V_i$  under the evaluation map.

We will show that there exists an  $m \gg 0$  such that the *m-Kempf filtration* of  $(E, \varphi)$  does not depend on  $m$ .

**Definition 6.7.** Let  $m \geq m_0$ . Given  $V_1 \subset \cdots \subset V_{t+1} = V$  a filtration of vector spaces of  $V$ . Let

$$v_{m,i} = m^{n+1} \cdot \frac{1}{\dim V^i \dim V} [r^i \dim V - r \dim V^i + \frac{a_2}{a_1} (\epsilon^i(\overline{\Phi}) \dim V - \dim V^i)],$$

$$b_m^i = \frac{1}{m^n} \dim V^i > 0$$

$$w_m^i = -b_m^i \cdot v_{m,i} = m \cdot \frac{1}{\dim V} [r \dim V^i - r^i \dim V + \frac{a_2}{a_1} (\dim V^i - \epsilon^i(\overline{\Phi}) \dim V)].$$

Also let

$$\begin{aligned} b_{m,i} &= b_m^1 + \cdots + b_m^i = \frac{1}{m^n} \dim V_i \\ w_{m,i} &= w_m^1 + \cdots + w_m^i = \\ &= m \cdot \frac{1}{\dim V} [r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (\dim V_i - \epsilon_i(\overline{\Phi}) \dim V)]. \end{aligned}$$

We call the graph defined by points  $(b_{m,i}, w_{m,i})$  the graph associated to the filtration  $V_\bullet \subset V$ .

Now, by Proposition 3.2 we can identify as well the new Kempf function in Theorem 1.4

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V) + \frac{a_2}{a_1} \sum_{i=1}^t n_i (\dim V_i - \epsilon_i(\overline{\Phi}) \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V_i \Gamma_i^2}},$$

with the function in Theorem 2.2, where the coordinates of the graph now are given in Definition 6.7.

Let us give the analogous to Propositions 3.8 and 3.10. Let

$$(6.5) \quad C = \max\{r|\mu_{\max}(E)| + \frac{d}{r} + r|B| + |A| + \delta_{n-1}(n-1)! + 1, 1\},$$

a positive constant, where  $\delta_{n-1}$  is the  $n-1$ -degree coefficient of the polynomial  $\delta(m)$  (if  $\deg(\delta) < n-1$ , then set  $\delta_{n-1} = 0$ ).

**Proposition 6.8.** *Given a sufficiently large  $m$ , each filter in the  $m$ -Kempf filtration of  $(E, \varphi)$  (cf. (6.4)) has slope  $\mu(E_i^m) \geq \frac{d}{r} - C$ .*

**Proof.** Choose an  $m_1$  such that for  $m \geq m_1$

$$[\mu_{\max}(E) + gm + B]_+ = \mu_{\max}(E) + gm + B$$

and

$$[\frac{d}{r} - C + gm + B]_+ = \frac{d}{r} - C + gm + B.$$

Let  $m_2$  be such that  $P_E(m) - \delta(m) > 0$  for  $m \geq m_2$ . Now consider  $m \geq \max\{m_0, m_1, m_2\}$  and let

$$0 \subsetneq E_1^m \subsetneq E_2^m \subsetneq \cdots \subsetneq E_t^m \subsetneq E_{t+1}^m = E$$

be the  $m$ -Kempf filtration.

Suppose we have a filter  $E_i^m \subseteq E$ , of rank  $r_i$  and degree  $d_i$ , such that  $\mu(E_i^m) < \frac{d}{r} - C$ . The subsheaf  $E_i^m(m) \subset E(m)$  satisfies the estimate in Lemma 3.6,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!} ((r_i - 1)([\mu_{\max}(E_i^m) + gm + B]_+)^n + ([\mu_{\min}(E_i^m) + gm + B]_+)^n),$$

where  $\mu_{\max}(E_i^m(m)) = \mu_{\max}(E_i^m) + gm$  and similarly for  $\mu_{\min}$ .

Note that  $\mu_{\max}(E_i^m) \leq \mu_{\max}(E)$  and  $\mu_{\min}(E_i^m) \leq \mu(E_i^m) < \frac{d}{r} - C$ , so

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!} ((r_i - 1)([\mu_{\max}(E) + gm + B]_+)^n + ([\frac{d}{r} - C + gm + B]_+)^n),$$

and, by choice of  $m$ ,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!} ((r_i - 1)(\mu_{\max}(E) + gm + B)^n + (\frac{d}{r} - C + gm + B)^n) = G(m),$$

where

$$G(m) = \frac{1}{g^{n-1}n!} [r_i g^n m^n + n g^{n-1} ((r_i - 1)\mu_{\max}(E) + \frac{d}{r} - C + r_i B) m^{n-1} + \cdots].$$

Recall that, by Definition 3.1, to the filtration (6.3) we associate a graph with heights, for each  $j$

$$w_j = w^1 + \cdots + w^j = m \cdot \frac{1}{\dim V} [r \dim V_j - r_j \dim V + \frac{a_2}{a_1} (\dim V_j - \epsilon_j(\overline{\Phi}) \dim V)].$$

We will show that  $w_i < 0$  and will get a contradiction as in Proposition 3.8. Since  $E_i^m(m)$  is generated by  $V_i$  under the evaluation map, it is  $\dim V_i \leq H^0(E_i^m(m))$ , hence

$$\begin{aligned} w_i &= \frac{m}{\dim V} [r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (\dim V_i - \epsilon_i(\overline{\Phi}) \dim V)] \leq \\ &= \frac{m}{P_E(m)} [r h^0(E_i^m(m)) - r_i P_E(m) + \frac{r \delta(m)}{P_E(m) - \delta(m)} (h^0(E_i^m(m)) - \epsilon_i(\overline{\Phi}) P_E(m))] \leq \\ &= \frac{m}{P_E(m)} [r G(m) - r_i P_E(m) + \frac{r \delta(m)}{P_E(m) - \delta(m)} (G(m) - \epsilon_i(\overline{\Phi}) P_E(m))] = \\ &= m \cdot \frac{[(P_E(m) - \delta(m))(r G(m) - r_i P_E(m)) + (r \delta(m))(G(m) - \epsilon_i(\overline{\Phi}) P_E(m))]}{P_E(m)(P_E(m) - \delta(m))}. \end{aligned}$$

Hence  $w_i < 0$  is equivalent to

$$\Psi(m) = (P_E(m) - \delta(m))(rG(m) - r_i P_E(m)) + (r\delta(m))(G(m) - \epsilon_i(\bar{\Phi})P_E(m)) < 0$$

and  $\Psi(m) = \xi_{2n}m^{2n} + \xi_{2n-1}m^{2n-1} + \dots + \xi_1m + \xi_0$  is a  $2n$ -order polynomial. Let us calculate the higher order coefficients:

$$\begin{aligned} \xi_{2n} &= (P_E(m) - \delta(m))_n(rG(m) - r_i P_E(m))_n + (r\delta(m))_n(G(m) - \epsilon_i(\bar{\Phi})P_E(m))_n = \\ &= (P_E(m) - \delta(m))_n(r\frac{r_i g}{n!} - r_i\frac{r g}{n!}) + 0 = 0. \end{aligned}$$

Then,  $\Psi(m)$  has no coefficient in order  $2n$ . Let us calculate the  $(2n-1)$ -coefficient:

$$\begin{aligned} \xi_{2n-1} &= (P_E(m) - \delta(m))_n(rG(m) - r_i P_E(m))_{n-1} + (r\delta(m))_{n-1}(G(m) - \epsilon_i(\bar{\Phi})P_E(m))_n = \\ &= \frac{r g}{n!}(rG_{n-1} - r_i\frac{A}{(n-1)!}) + r\delta_{n-1}(\frac{r_i g}{n!} - \epsilon_i(\bar{\Phi})\frac{r g}{n!}) \end{aligned}$$

where  $G_{n-1}$  is the  $n-1$ -coefficient of the polynomial  $G(m)$ ,

$$\begin{aligned} G_{n-1} &= \frac{1}{g^{n-1}n!}ng^{n-1}((r_i - 1)\mu_{\max}(E) + \frac{d}{r} - C + r_i B) = \\ &= \frac{1}{(n-1)!}((r_i - 1)\mu_{\max}(E) + \frac{d}{r} - C + r_i B) \leq \\ &= \frac{1}{(n-1)!}((r_i - 1)|\mu_{\max}(E)| + \frac{d}{r} - C + r_i |B|) \leq \\ &= \frac{1}{(n-1)!}(r|\mu_{\max}(E)| + \frac{d}{r} - C + r|B|) < \frac{-|A|}{(n-1)!} - \delta_{n-1}, \end{aligned}$$

last inequality coming from the definition of  $C$  in (6.5). Then

$$\begin{aligned} \xi_{2n-1} &< \frac{r g}{n!}(r(\frac{-|A|}{(n-1)!} - \delta_{n-1}) - r_i\frac{A}{(n-1)!}) + r\delta_{n-1}(\frac{r_i g}{n!} - \epsilon_i(\bar{\Phi})\frac{r g}{n!}) = \\ &= \frac{r g}{n!}[(\frac{-r|A| - r_i A}{(n-1)!}) - r\delta_{n-1} + \delta_{n-1}(r_i - \epsilon_i(\bar{\Phi})r)] = \\ &= \frac{r g}{n!}[(\frac{-r|A| - r_i A}{(n-1)!}) + \delta_{n-1}(r_i - (1 + \epsilon_i(\bar{\Phi}))r)] < 0 \end{aligned}$$

because  $-r|A| - r_i A < 0$ ,  $r_i - (1 + \epsilon_i(\bar{\Phi}))r < 0$  and  $\delta_{n-1} \geq 0$ . Note that if  $r_i = r$ , then  $\epsilon_i(\bar{\Phi}) = \epsilon_{t+1}(\bar{\Phi}) = 1$ .

Therefore  $\Psi(m) = \xi_{2n-1}m^{2n-1} + \dots + \xi_1m + \xi_0$  with  $\xi_{2n-1} < 0$ , so there exists  $m_3$  such that for  $m \geq m_3$  we will have  $\Psi(m) < 0$  and  $w_i < 0$ , then the contradiction. ■

Now we can prove the following proposition in a similar way as we proved Proposition 3.9.

**Proposition 6.9.** *There exists an integer  $m_4$  such that for  $m \geq m_4$  the sheaves  $E_i^m$  and  $E^{m,i} = E_i^m/E_{i-1}^m$  are  $m_4$ -regular. In particular their higher cohomology groups vanish and they are generated by global sections.*

**Proposition 6.10.** *Let  $m \geq m_4$ . For each filter  $E_i^m$  in the  $m$ -Kempf filtration of  $(E, \varphi)$ , we have  $\dim V_i = h^0(E_i^m)$ , therefore  $V_i \cong H^0(E_i^m)$ .*

**Proof.** Let  $V_\bullet \subseteq V$  be the Kempf filtration of  $V$  (cf. Theorem 1.4) and let  $E_\bullet^m \subseteq E$  be the  $m$ -Kempf filtration of  $(E, \varphi)$  (cf. (6.3) and (6.4)). We know that each  $V_i$  generates the subsheaf  $E_i^m$ , by definition, then we have the following diagram:

$$\begin{array}{ccccccc} 0 & \subset & V_1 & \subset & V_2 & \subset & \cdots \subset V_{t+1} = V \\ & & \cap & & \cap & & \parallel \\ & & H^0(E_1^m(m)) & \subset & H^0(E_2^m(m)) & \subset & \cdots \subset H^0(E_{t+1}^m(m)) = H^0(E(m)) \end{array}$$

Suppose there exists an index  $i$  such that  $V_i \neq H^0(E_i^m(m))$ . Let  $i$  be the index such that  $V_i \neq H^0(E_i^m(m))$  and  $\forall j > i$  it is  $V_j = H^0(E_j^m(m))$ . Then we have the diagram:

$$(6.6) \quad \begin{array}{ccc} V_i & \subset & V_{i+1} \\ \cap & & \parallel \\ H^0(E_i^m(m)) & \subset & H^0(E_{i+1}^m(m)) \end{array}$$

Therefore  $V_i \subsetneq H^0(E_i^m(m)) \subsetneq V_{i+1}$  and we can consider a new filtration by adding the filter  $H^0(E_i^m(m))$ :

$$(6.7) \quad \begin{array}{ccccc} V_i & \subset & H^0(E_i^m(m)) & \subset & V_{i+1} \\ \parallel & & \parallel & & \parallel \\ V'_i & & V'_{i+1} & & V'_{i+2} \end{array}$$

Note that  $V_i$  and  $H^0(E_i^m(m))$  generate the same sheaf  $E_i^m$ , hence we are in situation of Lemma 3.5, where  $W = H^0(E_i^m(m))$ , filtration  $V_\bullet$  is (6.6) and filtration  $V'_\bullet$  is (6.7).

The graph associated to filtration  $V_\bullet$ , by Definition 3.1, is given by the points

$$(b_i, w_i) = \left( \frac{\dim V_i}{m^n}, \frac{m}{\dim V} (r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (\dim V_i - \epsilon_i(\overline{\Phi}, V_\bullet) \dim V)) \right),$$

where the slopes of the graph are given by

$$\begin{aligned} -v_i &= \frac{w_i}{b_i} = \frac{w_i - w_{i-1}}{b_i - b_{i-1}} = \\ &= \frac{m^{n+1}}{\dim V} \left( r - r^i \frac{\dim V}{\dim V^i} + \frac{a_2}{a_1} (1 - \epsilon^i(\overline{\Phi}, V_\bullet) \frac{\dim V}{\dim V^i}) \right) \leq \\ &= \frac{m^{n+1}}{\dim V} \left( r + \frac{a_2}{a_1} \right) := R \end{aligned}$$

and equality holds if and only if  $r^i = 0$ . Here note that  $r^i = 0$  implies  $\epsilon^i(\overline{\Phi}, V_\bullet) = 0$ .

Now, the new point which appears in the graph of the filtration  $V'_\bullet$  is

$$Q = \left( \frac{h^0(E_i^m)}{m^n}, \frac{m}{\dim V} (r h^0(E_i^m) - r_i \dim V + \frac{a_2}{a_1} (h^0(E_i^m) - \epsilon_i(\overline{\Phi}, V_\bullet) \dim V)) \right),$$

where we write  $\epsilon_i(\overline{\Phi}, V_\bullet)$  instead of  $\epsilon_i(\overline{\Phi}, V'_\bullet)$ , because they are equal given that  $V_i = V'_i$ .

Point  $Q$  joins two new segments appearing in this new graph. The slope of the segment between  $(b_i, w_i)$  and  $Q$  is, by a similar calculation,

$$-v'_{i+1} = \frac{m^{n+1}}{\dim V} \left( r + \frac{a_2}{a_1} \right) = R.$$

By Lemma 3.4, the graph is convex, so  $v_1 < v_2 < \dots < v_{t+1}$ . As  $E_1^m$  is a non-zero torsion-free sheaf, it has positive rank  $r_1 = r^1$  and so it follows  $v_1 > -R$ .

Recall that, by definition,  $\epsilon_i(\overline{\Phi}, V_\bullet)$  is equal to 1 if  $\overline{\Phi}|_{V_i} \neq 0$  and 0 otherwise. Then, it is clear that

$$(6.8) \quad \begin{aligned} \epsilon_j(\overline{\Phi}, V'_\bullet) &= \epsilon_j(\overline{\Phi}, V_\bullet), j \leq i \\ \epsilon_j(\overline{\Phi}, V'_\bullet) &= \epsilon_{j-1}(\overline{\Phi}, V_\bullet), j > i \end{aligned}$$

and note  $\epsilon_i(\overline{\Phi}, V_\bullet) = \epsilon_{i+1}(\overline{\Phi}, V'_\bullet)$ . Then, the graph associated to  $V'_\bullet \subset V$  is a refinement of the graph associated to Kempf filtration  $V_\bullet \subset V$ , therefore by Lemma 3.5,  $v'_{i+1} \geq v_{i+1}$ . Hence

$$-R < v_1 < v_2 < \dots < v_{i+1} \leq v'_{i+1} = -R,$$

which is a contradiction.

Therefore,  $\dim V_i = h^0(E_i^m)$ , for every filter in the  $m$ -Kempf filtration. ■

**Corollary 6.11.** *For every filter  $E_i^m$  in the  $m$ -Kempf filtration of  $(E, \varphi)$ , it is  $r^i > 0$ .*

**Proof.** C.f. Corollary 3.11. ■

Now let us recall the results on section 4. By Proposition 6.9, for any  $m \geq m_4$ , all the filters  $E_i^m$  of the  $m$ -Kempf filtration of the pair  $(E, \varphi)$  are  $m_4$ -regular and hence, the filtration of sheaves

$$E_1^m \subset E_2^m \subset \dots \subset E_{t_m}^m \subset E_{t_m+1}^m = E$$

is the filtration associated to the filtration of vector subspaces

$$H^0(E_1^m(m_4)) \subset H^0(E_2^m(m_4)) \subset \dots \subset H^0(E_{t_m}^m(m_4)) \subset H^0(E_{t_m+1}^m(m_4)) = H^0(E(m_4))$$

by the evaluation map, of a unique vector space  $H^0(E(m_4))$ , whose dimension is independent of  $m$ . Let

$$(P_1^m, \dots, P_{t_m+1}^m)$$

the  $m$ -type of the  $m$ -Kempf filtration of  $(E, \varphi)$  and let

$$\mathcal{P} = \{(P_1^m, \dots, P_{t_m+1}^m)\}$$

set of possible  $m$ -types, which is a finite set (c.f. Proposition 4.1).

By Definition 6.7 we associate a graph to the  $m$ -Kempf filtration, given by  $v_m$ , which, thanks to Propositions 6.9 and 6.10, can be rewritten as

$$v_{m,i} = m^{n+1} \cdot \frac{1}{P_m^i(m)P(m)} \left[ r^i P(m) - r P_m^i(m) + \frac{r\delta(m)}{P(m) - \delta(m)} (\epsilon^i(\overline{\Phi})P(m) - P_m^i(m)) \right]$$

$$b_m^i = \frac{1}{m^n} \cdot P_m^i(m).$$

Let

$$v_{m,i}(l) = l^{n+1} \cdot \frac{1}{P_m^i(l)P(l)} \left[ r^i P(l) - r P_m^i(l) + \frac{r\delta(l)}{P(l) - \delta(l)} (\epsilon^i(\overline{\Phi})P(l) - P_m^i(l)) \right]$$

and define

$$\Theta_m(l) = (\mu_{v_m(l)}(\Gamma_{v_m(l)}))^2 = \|v_m(l)\|^2,$$

(c.f. (4.1)). Let  $\mathcal{A}$  be the finite set

$$\mathcal{A} = \{\Theta_m : m \geq m_4\},$$



and let  $K$  be a rational function such that there exists an integer  $m_5$  with  $\Theta_m = K$ ,  $\forall m \geq m_5$  (c.f. Lemma 4.2).

**Proposition 6.12.** *Let  $l_1$  and  $l_2$  be integers with  $l_1 \geq l_2 \geq m_5$ . Then the  $l_1$ -Kempf filtration of  $E$  is equal to the  $l_2$ -Kempf filtration of  $(E, \varphi)$ .*

**Proof.** C.f. Proposition 4.3. ■

**Definition 6.13.** *If  $m \geq m_5$ , the  $m$ -Kempf filtration of  $(E, \varphi)$  is called the Kempf filtration of  $(E, \varphi)$ ,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E.$$

## 7. HARDER-NARASIMHAN FILTRATION FOR HOLOMORPHIC PAIRS

Let  $m \geq m_5$ . Kempf's theorem (c.f. Theorem 1.4) asserts that given  $V \simeq H^0(E(m))$ , there exists a unique weighted filtration of vector spaces  $V_\bullet \subseteq V$  which gives maximum for the Kempf function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V} (r^i \dim V - r \dim V^i) + \frac{a_2}{a_1} \sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V} (\epsilon^i(\bar{\Phi}) \dim V - \dim V^i)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}}.$$

This filtration induces a filtration of sheaves, called the Kempf filtration of  $(E, \varphi)$ ,

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

which is independent of  $m$ , for  $m \geq m_5$ , by Proposition 6.12, hence it is unique.

We proceed in a similar way to section 5, to rewrite the Kempf function for pairs in terms of Hilbert polynomials of sheaves. Let  $\epsilon^i := \epsilon^i(\bar{\Phi}) = \epsilon^i(\varphi)$  and note that  $\epsilon^i = 1$  for the unique index  $i$  in the Kempf filtration such that  $\varphi|_{E_i} \neq 0$  and  $\varphi|_{E_{i-1}} = 0$ , and  $\epsilon^i = 0$  otherwise. Let us call this index  $j$  in the following.

**Proposition 7.1.** *Given a pair  $(E, \varphi : E \rightarrow \mathcal{O}_X)$ , there exists a unique filtration*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

*with positive weights  $n_1, \dots, n_t$ , which gives maximum for the function*

$$\mu = \frac{m^{\frac{n}{2}+1}}{P} \cdot \frac{\sum_{i=1}^{t+1} \gamma_i [(r^i P - r P^i) + \frac{r\delta}{P-\delta} (\epsilon^i P - P^i)]}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}}.$$

Similarly, we can express the function  $\mu$  in the proposition as

$$\mu = m^{-\frac{n}{2}} \cdot \frac{\sum_{i=1}^{t+1} P^i \gamma_i v_i}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}} = m^{-\frac{n}{2}} \cdot \frac{(\gamma, v)}{\|\gamma\|},$$

where the coordinates  $v_{i,m}$  (slopes of segments of the graph), now are

$$v_i = m^{n+1} \cdot \frac{1}{P^i P} \left[ r^i P - r P^i + \frac{r\delta}{P-\delta} (\epsilon^i P - P^i) \right]$$

and the scalar product is again

$$\begin{pmatrix} P^1 & & & \\ & P^2 & & \\ & & \ddots & \\ & & & P^{t+1} \end{pmatrix}$$

With Definition 6.3 the coordinates of the graph are

$$v_i = m^{n+1} \cdot \frac{r^i r}{P^i(P - \delta)} \left( \frac{\overline{P}(E)}{r} - \frac{\overline{P}(E^i)}{r^i} \right),$$

where recall the definition of the corrected Hilbert polynomial of the quotient  $(E^i, \varphi|_{E^i})$  (c.f. Definition 6.2),  $\overline{P}(E^i) = P^i - \delta \epsilon^i$ .

**Definition 7.2.** *Given a pair  $(E, \varphi : E \rightarrow \mathcal{O}_X)$ , a filtration*

$$0 \subset E_1 \subset E_2 \subset \dots \subset E_t \subset E_{t+1} = E$$

*is called a **Harder-Narasimhan filtration** of the pair if satisfies these two properties, where  $E^i = E_i/E_{i-1}$ ,*

(1) *The corrected Hilbert polynomials verify*

$$\frac{\overline{P}(E^1)}{\text{rk } E^1} > \frac{\overline{P}(E^2)}{\text{rk } E^2} > \dots > \frac{\overline{P}(E^{t+1})}{\text{rk } E^{t+1}}$$

(2) *Every block  $(E^i, \varphi|_{E^i})$  is semistable as a quotient pair.*

**Proposition 7.3.** *Given the Kempf filtration of a pair  $(E, \varphi)$ ,*

$$0 \subset E_1 \subset E_2 \subset \dots \subset E_t \subset E_{t+1} = E$$

*it verifies*

$$\frac{\overline{P}(E^1)}{\text{rk } E^1} > \frac{\overline{P}(E^2)}{\text{rk } E^2} > \dots > \frac{\overline{P}(E^{t+1})}{\text{rk } E^{t+1}}.$$

**Proof.** Let  $j$  be the unique index such that  $\epsilon^j = 1$ . By Lemma 3.4 it is  $v_1 < v_2 < \dots < v_{j-1} < v_j < v_{j+1} < \dots < v_{t+1}$ . Note that for  $i \neq j$  it is  $\overline{P}^i = P^i$ , hence  $v_{i-1} < v_i$  implies  $\frac{\overline{P}(E^{i-1})}{\text{rk } E^{i-1}} > \frac{\overline{P}(E^i)}{\text{rk } E^i}$  for all  $i \neq j, j+1$ .

Now the inequality  $v_{j-1} < v_j$  is

$$\frac{r^{j-1}r}{P^{j-1}(P - \delta)} \left( \frac{P - \delta}{r} - \frac{P^{j-1}}{r^{j-1}} \right) < \frac{r^j r}{P^j(P - \delta)} \left( \frac{P - \delta}{r} - \frac{P^j - \delta}{r^j} \right)$$

or, equivalently,

$$-\delta \frac{r P^{j-1}}{P - \delta} < P^{j-1} r^j - P^j r^{j-1}.$$

The function  $\frac{r P^{j-1}}{P - \delta}$  is a homogeneous rational function whose limit at infinity is  $r^{j-1}$ , so for large values of the variable we obtain this inequality between the polynomials

$$-\delta r^{j-1} < P^{j-1} r^j - P^j r^{j-1},$$

which is equivalent to  $\frac{\overline{P}(E^{j-1})}{\text{rk } E^{j-1}} > \frac{\overline{P}(E^j)}{\text{rk } E^j}$ . A similar argument proves that  $\frac{\overline{P}(E^j)}{\text{rk } E^j} > \frac{\overline{P}(E^{j+1})}{\text{rk } E^{j+1}}$ .

■

**Proposition 7.4.** *Given the Kempf filtration of a pair  $(E, \varphi)$ ,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

*each one of the blocks  $(E^i, \varphi|_{E^i})$  is semistable as a quotient pair.*

**Proof.** Suppose that any of the blocks has a destabilizing subpair and apply a similar argument to the one in Proposition 5.3. ■

Hence, having seen the properties of the Kempf filtration in Propositions 7.3 and 7.4, we have that the Kempf filtration of a pair  $(E, \varphi)$  is a Harder-Narasimhan filtration. We will prove that every pair has a unique Harder-Narasimhan filtration, therefore it will be the same that the Kempf filtration.

**Theorem 7.5.** *Every pair  $(E, \varphi)$  has a unique Harder-Narasimhan filtration.*

**Lemma 7.6.** *Let  $(E, \varphi)$  be a pair. Then, there exists a subsheaf  $F \subset E$  such that for all subsheaves  $G \subset E$ , one has  $\frac{\overline{P}(F)}{\text{rk } F} \geq \frac{\overline{P}(G)}{\text{rk } G}$ , and in case of equality  $G \subset F$ . Moreover,  $F$  is uniquely determined and  $(F, \varphi|_F)$  is  $\delta$ -semistable, called the maximal destabilizing subpair of  $(E, \varphi)$ .*

**Proof.** The last two assertions follow from the first, where note that being  $\delta$ -semistable can be checked by subpairs, by Lemma 6.4.

Define an order relation on the set of subpairs of  $(E, \varphi)$  by  $(F_1, \varphi|_{F_1}) \leq (F_2, \varphi|_{F_2})$  if and only if  $F_1 \subset F_2$  and  $\frac{\overline{P}(F_1)}{\text{rk } F_1} \leq \frac{\overline{P}(F_2)}{\text{rk } F_2}$ . Every ascending chain is bounded by  $(E, \varphi)$ , then by Zorn's Lemma, for every subpair  $(F, \varphi|_F)$  there exists a  $F \subset F' \subset E$  such that  $(F', \varphi|_{F'})$  is maximal with respect to  $\leq$ . Let  $(F, \varphi|_F)$  be  $\leq$ -maximal with  $F$  of minimal rank among all maximal subpairs. We claim that  $F$  has the asserted properties.

Suppose there exists  $G \subset E$  with  $\frac{\overline{P}(G)}{\text{rk } G} \geq \frac{\overline{P}(F)}{\text{rk } F}$ . First, we show that we can assume  $G \subset F$  by replacing  $G$  by  $G \cap F$ . Indeed, if  $G \not\subset F$ , then  $F$  is a proper subsheaf of  $F + G$  and hence  $\frac{\overline{P}(F)}{\text{rk } F} > \frac{\overline{P}(F+G)}{\text{rk } F+G}$ , by definition of  $F$ . Using the exact sequence

$$0 \rightarrow F \cap G \rightarrow F \oplus G \rightarrow F + G \rightarrow 0$$

one finds

$$P(F) + P(G) = P(F \oplus G) = P(F \cap G) + P(F + G)$$

and

$$\text{rk}(F) + \text{rk}(G) = \text{rk}(F \oplus G) = \text{rk}(F \cap G) + \text{rk}(F + G).$$

Calculating we have

$$\text{rk}(F \cap G) \left( \frac{P(G)}{\text{rk } G} - \frac{P(F \cap G)}{\text{rk}(F \cap G)} \right) = \text{rk}(F + G) \left( \frac{P(F + G)}{\text{rk}(F + G)} - \frac{P(F)}{\text{rk } F} \right) + (\text{rk}(G) - \text{rk}(F \cap G)) \left( \frac{P(F)}{\text{rk } F} - \frac{P(G)}{\text{rk } G} \right).$$

Using

$$\epsilon(F \cap G) + \epsilon(F + G) \leq \epsilon(F) + \epsilon(G)$$

we get

$$(P(F) - \delta\epsilon(F)) + (P(G) - \delta\epsilon(G)) = (P(F \cap G) - \delta\epsilon(F \cap G)) + (P(F + G) - \delta\epsilon(F + G))$$

and similarly,

$$\text{rk}(F \cap G) \left( \frac{\overline{P}(G)}{\text{rk } G} - \frac{\overline{P}(F \cap G)}{\text{rk}(F \cap G)} \right) \leq \text{rk}(F + G) \left( \frac{\overline{P}(F + G)}{\text{rk}(F + G)} - \frac{\overline{P}(F)}{\text{rk } F} \right) + (\text{rk}(G) - \text{rk}(F \cap G)) \left( \frac{\overline{P}(F)}{\text{rk } F} - \frac{\overline{P}(G)}{\text{rk } G} \right).$$

Then, together with the two inequalities  $\frac{\overline{P}(F)}{\text{rk } F} \leq \frac{\overline{P}(G)}{\text{rk } G}$  and  $\frac{\overline{P}(F)}{\text{rk } F} > \frac{\overline{P}(F+G)}{\text{rk}(F+G)}$  we obtain

$$\frac{\overline{P}(G)}{\text{rk } G} - \frac{\overline{P}(F \cap G)}{\text{rk}(F \cap G)} < 0$$

and hence

$$\frac{\overline{P}(F)}{\text{rk } F} < \frac{\overline{P}(F \cap G)}{\text{rk}(F \cap G)}.$$

Next, fix  $G \subset F$  with  $\frac{\overline{P}(G)}{\text{rk } G} > \frac{\overline{P}(F)}{\text{rk } F}$  such that  $(G, \varphi|_G)$  is  $\leq$ -maximal in  $(F, \varphi|_F)$ . Then let  $(G', \varphi|_{G'}) \geq (G, \varphi|_G)$ ,  $\leq$ -maximal in  $(E, \varphi)$ . In particular,  $\frac{\overline{P}(F)}{\text{rk } F} < \frac{\overline{P}(G)}{\text{rk } G} \leq \frac{\overline{P}(G')}{\text{rk } G'}$ . By maximality of  $(G', \varphi|_{G'})$  and  $(F, \varphi|_F)$  we know  $G' \not\subset F$ , since otherwise  $\text{rk}(G') < \text{rk}(F)$  contradicting the minimality of  $\text{rk}(F)$ . Hence,  $F$  is a proper subsheaf of  $F + G'$ . Therefore,  $\frac{\overline{P}(F)}{\text{rk } F} > \frac{\overline{P}(F+G')}{\text{rk}(F+G')}$ . As before the inequalities  $\frac{\overline{P}(F)}{\text{rk } F} < \frac{\overline{P}(G')}{\text{rk } G'}$  and  $\frac{\overline{P}(F)}{\text{rk } F} > \frac{\overline{P}(F+G')}{\text{rk}(F+G')}$  imply

$$\frac{\overline{P}(F \cap G')}{\text{rk}(F \cap G')} > \frac{\overline{P}(G')}{\text{rk } G'} \geq \frac{\overline{P}(G)}{\text{rk } G}.$$

Since  $G \subset F \cap G' \subset F$ , this contradicts the assumption on  $G$ . ■

**Proof of the Theorem.** The Lemma allows to prove the existence of a Harder-Narasimhan filtration for  $(E, \varphi)$ . Let  $(E_1, \varphi|_{E_1})$  the maximal destabilizing subpair and suppose that the corresponding quotient  $(E/E_1, \varphi|_{E/E_1})$  has a Harder-Narasimhan filtration,

$$0 \subset G_0 \subset G_1 \subset \dots \subset G_{t-1} = E/E_1,$$

by induction. We define  $E_{i+1}$  the pre-image of  $G_1$  and it is  $\frac{\overline{P}(E_1)}{\text{rk } E_1} > \frac{\overline{P}(E_2/E_1)}{\text{rk } E_2/E_1}$  because, if not, we would have  $\frac{\overline{P}(E_1)}{\text{rk } E_1} \leq \frac{\overline{P}(E_2)}{\text{rk } E_2}$ , contradicting the maximality of  $(E_1, \varphi|_{E_1})$ .

For the uniqueness, assume that  $E_\bullet$  and  $E'_\bullet$  are two Harder-Narasimhan filtrations. We consider, without loss of generality,  $\frac{\overline{P}(E'_1)}{\text{rk } E'_1} \geq \frac{\overline{P}(E_1)}{\text{rk } E_1}$ . Let  $j$  be minimal with  $E'_1 \subset E_j$ . Then the composition

$$E'_1 \rightarrow E_j \rightarrow E_j/E_{j-1}$$

is a non-trivial homomorphism of semistable sheaves. This implies

$$\frac{\overline{P}(E_j/E_{j-1})}{\text{rk } E_j/E_{j-1}} \geq \frac{\overline{P}(E'_1)}{\text{rk } E'_1} \geq \frac{\overline{P}(E_1)}{\text{rk } E_1} \geq \frac{\overline{P}(E_j/E_{j-1})}{\text{rk } E_j/E_{j-1}}$$

where first inequality comes from the fact that if there exists a non-trivial homomorphism between semistable pairs, then the corrected polynomial of the target is greater or equal than the one of the first pair. Hence, equality holds everywhere, implying  $j = 1$  so that  $E'_1 \subset E_1$ . Then, by semistability of  $E_1$ , it is  $\frac{\overline{P}(E'_1)}{\text{rk } E'_1} \leq \frac{\overline{P}(E_1)}{\text{rk } E_1}$ , and we can repeat the argument interchanging the roles of  $E_1$  and  $E'_1$  to show  $E_1 = E'_1$ . By induction we can assume that uniqueness holds for the Harder-Narasimhan filtrations of  $E/E_1$ . This shows  $E'_i/E_1 = E_i/E_1$  and finishes the proof. ■

**Corollary 7.7.** *Let  $(E, \varphi)$  a unstable pair. The Kempf filtration is the same that the Harder-Narasimhan filtration.*

**Proof.** By propositions 5.2 and 5.3 the Kempf filtration is a Harder-Narasimhan filtration, which is unique by Theorem 7.5, hence both filtrations are the same. ■

## REFERENCES

- [AB] M. F. Atiyah and R. Bott *The Yang Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. Ser. A **308** (1982), no. 1505, 523–615.
- [Br] L. Bruasse, *Optimal destabilizing vectors in some gauge theoretical moduli problems*. Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1805–1826
- [BT] L. Bruasse and A. Teleman *Harder-Narasimhan filtrations and optimal destabilizing vectors in complex geometry*. Ann. Inst. Fourier (Grenoble) **55** (2005), no. 3, 1017–1053.
- [Gi] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. Math., **106** (1977), 45–60.
- [He] W. H. Hesselink *Uniform instability in reductive groups*, J. Reine Angew. Math. **304** (1978), 74–96.
- [HK] V. Hoskins and F. Kirwan, *Quotients of unstable subvarieties and moduli spaces of sheaves of Fixed Harder-Narasimhan type*, [arXiv:1103.4731](#) (preprint 2011).
- [HL0] D. Huybrechts and M. Lehn, *Stable pairs on curves and surfaces*, J. Alg. Geom., **4** no. 1 (1995), 67–104.
- [HL1] D. Huybrechts and M. Lehn, *Framed modules and their moduli*, Intern. J. Math., **6** no. 2 (1995), 297–324.
- [HL2] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics E31, Vieweg, Braunschweig/Wiesbaden 1997.
- [Ke] G. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** no. 1 (1978), 299–316.
- [Ma] M. Maruyama, *Moduli of stable sheaves, I and II*. J. Math. Kyoto Univ. **17** (1977), 91–126. **18** (1978), 557–614.
- [GIT] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34. Springer-Verlag, Berlin, 1994.
- [Ne] P.E. Newstead, *Lectures on Introduction to Moduli Problems and Orbit Spaces*, Published for the Tata Institute of Fundamental Research, Bombay. Springer-Verlag, Berlin (1978).
- [Sa] F. Sala, *Restriction theorems for  $\mu$ -(semi)stable framed sheaves*, [arXiv:1010.5096](#) (preprint 2010).
- [Si] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publ. Math. I.H.E.S. **79** (1994), 47–129.

TG, AZ: INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM), NICOLÁS CABRERA 13-15, CAMPUS CANTOBLANCO UAM, 28049 MADRID, SPAIN

IS, AZ: DEPARTAMENTO DE ÁLGEBRA, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID, SPAIN

*E-mail address:* [tomas.gomez@icmat.es](mailto:tomas.gomez@icmat.es), [isols@mat.ucm.es](mailto:isols@mat.ucm.es), [alfonsozamora@icmat.es](mailto:alfonsozamora@icmat.es)